Semantic Pollution: Explicit, Implicit or Structural?

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Semantic Pollution and Purity of Methods

Assumption (An internal criterion for proof systems)

- *The rules should be capable of conferring meaning to the logical terms;*
- *Analytical proofs:*
  - *Nothing external to the content of the theorem should be used in the proof;*

Assumption (Semantic Pollution)

- ***The proof-system should not incorporate extraneous semantic elements*** *(syntactic purity);*
- *Semantic pollution makes unacceptable a proof system that follows all the other criteria.*

Observation (Purity of Methods)

Internal criteria supplemented with syntactic purity want to provide a formal interpretation of purity of methods applied to inferential systems.

Assumption (Criteria for Sequent Calculi)

Separation: *The rules for each connective/modality should be* ***independent of any other connective/modality****;*

Weak Symmetry: *Each rule should* ***either be a left or a right rule****.*

Symmetry ***Both*** *left and right rules for each connective/modality.*

Weak Explicitness (resp. explicitness): *The* ***connective/modality*** *appears* ***only*** *(resp. only and once) in the* ***conclusion*** *of the rule;*

Primitive interderivable modalities: *The two modalities* $\Box$ *and* $\Diamond$ *should be both* ***primitive*** *but* ***interderivable****;*

Uniqueness: *Each connective should be* ***uniquely characterized*** *by its rules in a given system;*

Došen's Principle: ***Different systems*** *are obtained by* ***changing only the structural rules****, while leaving the logical rules unaltered;*

Cut Elimination: *Cut should be* ***eliminable****;*

Subformula Property: *For every provable sequent, there is a proof in which only* ***subformulas*** *of the* ***formulas*** *in the* ***endsequent*** *occur.*

(Wansing (2002) pg. 9-10 and Negri, S. (2007))

Observation (Requirements for rules)

Separation: *The rules for each connective/modality should be* ***independent of any other connective/modality****;*

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Negri's Modal Systems and Explicit Pollution

Modal logic G3K

$$\Gamma, wRo \Rightarrow wRo, \Delta \qquad \Gamma, w : p \Rightarrow w : p, \Delta$$

$$\Box \Rightarrow \frac{\Gamma, o : A, w : \Box A, wRo \;\Rightarrow\; \Delta}{\Gamma, w : \Box A, wRo \;\Rightarrow\; \Delta} \qquad \Rightarrow \Box \frac{\Gamma, wRo \;\Rightarrow\; o : A, \Delta}{\Gamma \;\Rightarrow\; w : \Box A, \Delta}$$

$$\Diamond \Rightarrow \frac{\Gamma, o : A, wRo \;\Rightarrow\; \Delta}{\Gamma, w : \Diamond A \;\Rightarrow\; \Delta} \qquad \Rightarrow \Diamond \frac{\Gamma, wRo \;\Rightarrow\; w : \Diamond A, o : A, \Delta}{\Gamma, wRo \;\Rightarrow\; w : \Diamond A, \Delta}$$

$$Weak \Rightarrow \frac{\Gamma \;\Rightarrow\; \Delta}{\Gamma, w : A \;\Rightarrow\; \Delta} \qquad \Rightarrow Weak \frac{\Gamma \;\Rightarrow\; \Delta}{\Gamma \;\Rightarrow\; w : A, \Delta}$$

$$Weak \Rightarrow \frac{\Gamma \;\Rightarrow\; \Delta}{\Gamma, wRo \;\Rightarrow\; \Delta} \qquad \Rightarrow Weak \frac{\Gamma \;\Rightarrow\; \Delta}{\Gamma \;\Rightarrow\; wRo, \Delta}$$

$$Cut \frac{\Gamma \;\Rightarrow\; w : A, \Delta \qquad \Theta, w : A \;\Rightarrow\; \Lambda}{\Gamma, \Theta \;\Rightarrow\; \Delta, \Lambda}$$

$$\wedge \Rightarrow \frac{\Gamma, w : A, , w : B \;\Rightarrow\; \Delta}{\Gamma, w : A \wedge B \;\Rightarrow\; \Delta} \qquad \wedge \Rightarrow \frac{\Gamma, w : B \;\Rightarrow\; \Delta}{\Gamma, w : A \wedge B \;\Rightarrow\; \Delta}$$

Observation

In $\Box R$ and $\Diamond L$ o is a new label.

Example (Rules for accessibility)

$$\frac{wRw, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\ \text{Ref} \qquad \frac{wRr, wRo, oRr, \Gamma \Rightarrow \Delta}{wRo, oRr, \Gamma \Rightarrow \Delta}\ \text{Trans} \qquad \frac{oRw, wRo, \Gamma \Rightarrow \Delta}{wRo, \Gamma \Rightarrow \Delta}\ \text{Sym}$$

Observation

$$\begin{aligned} \textit{G3T} &= \textit{G3K} + \textit{Ref} \\ \textit{G3S4} &= \textit{G3K} + \textit{Ref} + \textit{Trans} \\ \textit{G3S5} &= \textit{G3K} + \textit{Ref} + \textit{Trans} + \textit{Sym} \end{aligned}$$

Points in favor

Observation (Good Properties)

- *Cut elimination;*
- *Height-preserving admissibility of structural rules;*
- *Height-preserving invertibility of operational rules;*
- *Axiom is already atomic.*
- *These good properties are preserved in the extensions.*

Observation (Došen's Principle and R)

Negri argues that the system suit Došen's Principle, since the different modal systems can be obtained by changing 'structural' rules for the accessibility relation.

Objection (Problems)

Negri's system has problems that are independent of semantic pollution, but let us ignore this issue for the moment.

Semantic pollution!

Objection (Elements from Kripke's Frames)

Negri's modal systems is semantically polluted since elements of Kripke's structure occur in it.

Objection (What's wrong with Semantic Pollution)

Poggiolesi: semantically polluted systems should be avoided because:

- *Semantically polluted systems do not operate with formulas and inferences, but with truth values and variables ranging over possible worlds, so the criteria of inferential acceptability lose their point (especially Došen's Principle (II) for logicality and inferential definition of the meaning of terms (III)).*

Objection (Irreflexive R)

Negri's system can describe accessibility relations that does not correspond to any modal logic, like irreflexive accessibility relations.

Theorem

*The extension of **G3K** with the rule* $\dfrac{}{wRw, \Gamma \Rightarrow \Delta}\,\textit{Irreflex}$ *is conservative.*

Answer (Good Proof Analysis)

Negri's system enables a easy and elegant proof:

> *Suppose that the sequent $\Gamma \Rightarrow \Delta$ (not containing uRv) is derivable in **G3K** + **Irref**. The atoms of the form xRy that appear on the left-hand side of sequents in the derivation originate from applications of rule $\Rightarrow \Box$. By the variable condition, $x \neq y$, so the derivation contains no atom of the form xRx, hence no application of **Irref**. Therefore the sequent is derivable in **G3K**.*

Theorem 7.1 of Negri, Proof Analysis in Modal Logic, 2005.

Observation (Good proof-analysis. Theory of meaning?)

Negri's system is good for proof analysis, is it problematic as a theory of meaning?

Semantic pollution?

Objection (Poggiolesi's *criterion* for semantic pollution: translation!)

"a sequent (...) does not contain a semantic element if every element that serves to define the sequent (...) may be translated in such a way that it forms , together with the translation of the other elements, a formula equivalent to the sequent."(p. 31)

$$\Gamma \Rightarrow \Delta \textit{ iff } \bigwedge \Gamma \supset \bigvee \Delta$$

Answer (Translating theoretical and semantic terms)

- *Negri's entire project is grounded on the idea of translating axiomatic sentences into sequent calculus;*
- *Objecting that the sentences that translate the sequents contain semantic elements is circular.*

Answer (inferentially acceptable = non (model theoretic) semantic)

Read: *Usage of labels is acceptable if the rules of the system are harmonious (suit PTS restrictions), like for mathematical theories.*

Objection (Non-Separability: Meaning-dependence)

Even accepting Read's point, Negri's rules establish a meaning dependence of modal rules over R-rules (as opposed to mathematical rules). (this is the reason why Read consider these rules structural!)

Answer (Accepting meaning-dependence)

$$\frac{\begin{array}{c}[A]\\ \bot\end{array}}{\neg A}\neg I \quad \rightsquigarrow \quad \bot \prec \neg \qquad \frac{\begin{array}{c}[A]\\ \vdots\\ B \vee D\end{array}}{(A \supset B) \vee D}\supset I_{Mln} \quad \rightsquigarrow \quad \vee \prec \supset$$

Observation (Upgrade of the requirements!)

We need to extend the requirement on accessibility rules:

From *Axiom extensions should not break the desiderata for proof systems: Cut-elimination, subformula, etc. should still hold for the **logical rules** of the system;*

To *If labels are used to confer meaning to logical terms, they should suit the desiderata for proof systems!*

Objection (No-Symmetry (I or E-rules))

Natural deduction rules are simultaneously (general)I and (general)E-rules:

$$\dfrac{\begin{matrix}[wRw]\\ \vdots \\ u:A\end{matrix}}{u:A}\;T\ (Refl) \qquad \dfrac{wRv \qquad vRu \qquad \begin{matrix}[wRu]\\ \vdots \\ t:A\end{matrix}}{t:A}\;4\ (Trans) \qquad \dfrac{wRv \qquad \begin{matrix}[vRw]\\ \vdots \\ t:A\end{matrix}}{t:A}\;B\ (Sym)$$

Even more obvious with the rules:

$$\dfrac{}{wRw}\;T\ (Refl) \qquad \dfrac{wRv \qquad vRu}{wRu}\;4\ (Trans) \qquad \dfrac{wRv}{vRw}\;B\ (Symm)$$

Objection (Harmony?)

The maximal formula

$$\dfrac{\dfrac{uRw}{wRu}\;B \qquad uRv}{wRv}\;4$$

cannot be obtained only because of the restriction on the minor premises of Read's rules.

- *Read interprets harmony as functionality of E-rules upon I-rules!*

Objection (No-Symmetry (Left or Right-rules))

Negri's rules are in some sense Left-rules, since operate with left hand side of $\Rightarrow$. However, they remove sentences instead of introducing them.

$$\frac{wRw, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\,\textit{Ref} \qquad \frac{wRr, wRo, oRr, \Gamma \Rightarrow \Delta}{wRo, oRr, \Gamma \Rightarrow \Delta}\,\textit{Trans}$$

$$\frac{oRw, wRo, \Gamma \Rightarrow \Delta}{wRo, \Gamma \Rightarrow \Delta}\,\textit{Sym} \qquad \frac{wRr, wRo, oRr, \Gamma \Rightarrow \Delta}{wRo, wRr, \Gamma \Rightarrow \Delta}\,\textit{Eucl}$$

Objection (Cut-elimination?)

Using groundsequents for the assumptions about accessibility relations, we end up with Cuts, while using Negri's rules these can be avoided:

$$\frac{\dfrac{}{uRw \Rightarrow wRu}\,\textit{Sym} \qquad \dfrac{}{wRu, uRv \Rightarrow wRv}\,\textit{Trans}}{uRw, uRv \Rightarrow wRv}\,\textit{Cut} \quad \sim \quad \frac{\dfrac{\dfrac{}{uRw, wRu, uRv, wRv \Rightarrow wRv}\,\textit{Axiom}}{uRw, wRu, uRv \Rightarrow wRv}\,\textit{Trans}}{uRw, uRv \Rightarrow wRv}\,\textit{Sym}$$

Negri's rules delete formulas from the premises because they absorb Cut:

$$\frac{\dfrac{}{wRo \Rightarrow oRw}\,\textit{Sym} \qquad wRo, oRw, \Gamma \Rightarrow \Delta}{wRo, \Gamma \Rightarrow \Delta}\,\textit{Cut} \quad \sim \quad \frac{wRo, oRw, \Gamma \Rightarrow \Delta}{wRo, \Gamma \Rightarrow \Delta}\,\textit{Sym}$$

Skip

Objection (Contraction is not really admissible)

When Ref is not present, we have to assume a contracted version of the $R-rules$ with more than one formula for accessibility in the conclusion.

$$\frac{wRw, wRw, wRw, \Gamma \Rightarrow \Delta}{wRw, wRw, \Gamma \Rightarrow \Delta}\,\textit{Trans} \qquad \frac{wRw, wRw, \Gamma \Rightarrow \Delta}{wRw, \Gamma \Rightarrow \Delta}\,\textit{Trans*}$$

Answer (Proof-search (Negri) and atomic contraction)

- *Contracted rules do not cause troubles for proof-search, since there is a bounded number of possible cases of contracted rules to be added.*
- *The only ineliminable contractions regard relational atoms, so the meaning of logical terms is independent of Contraction (Hacking).*

Example (Non-atomic weakening for S4)

In Ohinshi and Matsumoto's system

$$\frac{\Gamma, A \Rightarrow \Delta}{\Gamma, \Box A \Rightarrow \Delta}\,\text{L}\Box \qquad \frac{\Box\Gamma \Rightarrow A}{\Box\Gamma \Rightarrow \Box A}\,\text{R}\Box$$

weakening cannot be reduced to its atomic form: consider $\Box A \Rightarrow \Box A, \Box B$.

Objection (Lack of subformula property)

Negri's systems do not satisfy the subformula property, relational atoms occur in a Cut-free derivation together with the subformulas of the formulas in the endsequent.

Answer (Subterm property)

All terms in minimal derivations are terms found in the endsequent.

Observation (Proof-Analysis and PT-justification)

- *This is enough for proof-search and decidability (Negri).*
- *Is it enough for **analytic** proofs?*

Problems

Objection (Problems)

- *Subformula property doesn't hold;*
- *Closure Condition asks for the extension with extra rules when Refl is absent (to have Contraction admissible);*
- *Rules for* $\Box$ *and* $\Diamond$ *are separable, but both depend on* R*-rules;*
- *Symmetry and Explicitness hold for* $\Box$ *and* $\Diamond$*-rules but not for* R*-rules:*
  - *The rules are not properly L or R-rules;*
  - R *occur both in the conclusion and in the premises;*
- R*-rules are structural? (Does Došen principle hold?)*
- ***The calculus is semantically polluted by the reference to Kripke's frames.***

Poggiolesi's Modal Systems and Implicit Pollution

Syntactically Pure Systems

Example (Tree-hypersequents)

Poggiolesi proposes a system that preserves the tree-structure Kripke's frames but does not use explicit labels:

$$\frac{G\{\Box A, M \Rightarrow N, [A, P \Rightarrow Q]\}}{G\{\Box A, M \Rightarrow N, [P \Rightarrow Q]\}}\ \Box\text{L} \quad \sim \quad \frac{\Gamma, wRu \Rightarrow u : A, \Delta}{\Gamma, w : \Box A \Rightarrow \Delta}\ \Box\text{L}$$

$$\frac{G\{M \Rightarrow N, [\Rightarrow A]\}}{G\{M \Rightarrow N, \Box A\}}\ \Box\text{R} \quad \sim \quad \frac{\Gamma, wRo \ \Rightarrow \ o : A, \Delta}{\Gamma \ \Rightarrow \ w : \Box A, \Delta}\ \Box\text{R}$$

Observation (Subformula and Contraction)

- *Poggiolesi's systems satisfy the subformula property;*
- *Contraction is admissible in Poggiolesi's system, without need for the Closure Condition;*
- *It is syntactically pure (no labels).*

Not Explicitly Polluted

Assumption (It respects Poggiolesi's criterion)

Poggiolesi's systems suit the criterion:

> *... every element that serves to define the sequent (or set of sequents) may be translated in such a way that it forms, together with the translation of the other elements, a formula equivalent to the sequent. (p. 31)*

- $(\Gamma \Rightarrow \Delta)^\tau := \bigwedge \Gamma \supset \bigvee \Delta$;
- $(\Gamma \Rightarrow \Delta / G_1; ...; G_n)^\tau := (\Gamma \Rightarrow \Delta)^\tau \vee \Box G_1^\tau \vee ... \vee \Box G_n^\tau$

Implicit Semantic Pollution

Observation (Implicit Pollution)

According to Avron, a calculus can be semantically polluted even if no semantical label occur in it:

> *Because of the proof-theoretical nature and the expected generality, the framework should be independent of any particular semantics.* ***One should not be able to guess, just from the form of the structures which are used, the intended semantics of a given proof system.***

Avron, The method of hypersequents, 1996.

Objection (Poggiolesi's systems are (implicitly) semantically polluted)

- *They can be translated into Negri's systems;*
- *They are inspired by Kripke's frames.*

Answer (Poggiolesi)

- *Even some sequents for classical logic are (implicitly) semantically polluted;*
- *Inferential interpretation of tree-hypersequents.*

Poggiolesi's reduction

Objection (Poggiolesi-Read)

Kleene's context-sharing classical sequent calculi are isomorphic to tableaux-systems, which are regarded as clearly semantically inspired (even 'hybrids' between syntactic and semantic systems).

$$\dfrac{\dfrac{\dfrac{p \Rightarrow p, \neg q}{\Rightarrow p, \neg p, \neg q} \qquad \dfrac{q \Rightarrow q, \neg p}{\Rightarrow q, \neg p, \neg q}}{\dfrac{\Rightarrow p \wedge q, \neg p, \neg q}{\Rightarrow p \wedge q, \neg p \vee \neg q}}}{\neg(p \wedge q) \Rightarrow \neg p \vee \neg q}$$

$$\begin{array}{c} T\neg(p \wedge q) \\ F\neg p \vee \neg q \\ Fp \wedge q \\ F\neg p \\ F\neg q \\ Fp \mid Fq \\ Fq \mid Tq \\ \times \quad \times \end{array}$$

So, even sequent systems for classical logic should count as semantically polluted. Hence, ***implicit semantic pollution is too broadly applicable to be a good criterion****.*

Observation (Inferential Interpretation of Hypersequents)

Poggiolesi and Restall propose an inferential interpretation of tree-hypersequents;

> *Suppose $\Diamond(A \vee B)$. So,* ***in some circumstance****, $A \vee B$. There are two cases: Case (i) A, and Case (ii) B. Take case (i) first. Then in this circumstance, A and so,* ***back where we started****, $\Diamond A$, and hence $\Diamond A \vee \Diamond B$. On the other hand, we might have case (ii). There we have B, so back* ***in the original circumstance****, $\Diamond B$, and hence, $\Diamond A \vee \Diamond B$. So, in either case, we have $\Diamond A \vee \Diamond B$, which is what we wanted.*

The idea is to interpret modal logic as speaking of ***different circumstances****, delimited by* ***slashes*** *in tree-hypersequents.*

Objection (Same Interpretation for Negri's system?)

The same interpretation could be used for Negri's system!

Objection (There could be unicorns)

The problem is whether such an antirealistic interpretation holds for any modal logic. Dummett argues against symmetry of the accessibility relation between worlds, based on such an interpretation of modal logic.

Observation (Inferential Reading of Classical Logic)

Also for classical logic, semantically polluted systems could be read inferentially:

- *Bilateral (assertion/rejection) reading of tableaux-systems;*
  - *Bilateral reading of sequent calculus (Restall);*
- *In tableaux-systems, **T** can be omitted and **F** can be converted in* $\neg$.

Objection (Two Directions of Pollution)

Pollution goes in both directions:

- *an inferential reading of a system can be extended to any isomorphic system;*
- *a semantic (model-theoretic) reading of a system can be extended to any isomorphic system.*

Observation (External and Internal Criterion)

There are two alternatives:

- *An external criterion of semantic pollution should take into consideration non-formal meaning theoretical issues (Dummett);*
- *An internal criterion should not focus on the relationship with models!*

skip

External Criterions

Assumption (Logic and natural language)

The rules of logic should be understandable from the point of view of the general usage of language (to understand a game you must understand the goal too).

$$\frac{A \mapsto (B \vee C)}{(A \mapsto B) \vee (A \mapsto C)}$$

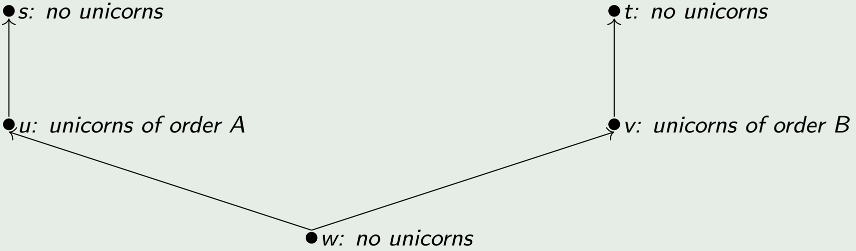
Observation (Criteria based on theory of meaning)

$$\dfrac{\dfrac{[A]^1}{A, \bot}\ \textit{Weakening}}{A, A \supset \bot}\ \supset I_1 \qquad \dfrac{\dfrac{\dfrac{[-(A \vee \neg A)]^1}{-A}\ - \vee E \qquad \dfrac{\dfrac{[-(A \vee \neg A)]^1}{-(\neg A)}\ - \vee E}{+A}\ - \neg E}{\bot}\ \textit{Non-contr}}{+(A \vee \neg A)}\ \textit{Reductio}\ _1$$

- *Multiple conclusion are defined circularly using (classical) disjunction;*
- *Rejection should be interpreted as assertion of a negation.*

Unicorns and symmetry (axiom B)

Argument (From unicorns to non-symmetry)



Argument (There could be unicorns)

Dummett rejects Kripke's argument that there could not be unicorns because it relies on presuppositions that are unacceptable for a semantic antirealist:

- *An initial baptism (or lack of) that is not accessible anymore to the community of speakers cannot be decisive to the correctness of the usage of a term.*

A non-formal (external to the proof-system) meaning-theoretical argument against Axiom B.

Degrees of Semantic Pollution for Classical Logic

LK and G4: a case study

LK	**G4**
$\dfrac{}{\vdash\ p, \overline{p}}$ ax	$\dfrac{}{\vdash\ \Gamma, p, \overline{p}}$ ax
$\dfrac{\vdash\ \Gamma, A \qquad \vdash\ \Delta, \overline{A}}{\vdash\ \Gamma, \Delta}$ Cut	$\dfrac{\vdash\ \Gamma, A \qquad \vdash\ \Gamma, \overline{A}}{\vdash\ \Gamma}$ Cut
$\dfrac{\vdash\ \Gamma, A}{\vdash\ \Gamma, A \vee B}\vee \qquad \dfrac{\vdash\ \Gamma, A}{\vdash\ \Gamma, A \vee B}\vee$	$\dfrac{\vdash\ \Gamma, A, B}{\vdash\ \Gamma, A \vee B}\vee$
$\dfrac{\vdash\ \Gamma, A \qquad \vdash\ \Delta, B}{\vdash\ \Gamma, A \wedge B, \Delta}\wedge$	$\dfrac{\vdash\ \Gamma, A \qquad \vdash\ \Gamma, B}{\vdash\ \Gamma, A \wedge B}\wedge$
$\dfrac{\vdash\ \Gamma}{\vdash\ \Gamma, A}$ W	$\dfrac{\vdash\ \Gamma, A, A}{\vdash\ \Gamma, A}$ C

Assumption (Fully atomic axioms)

All the formulas that occur in the axiom must be atomic.

Observation ('Packing' or 'unpacking' structural rules)

Converting derivations from LK to G4 and vice-versa is simply a matter of 'packing' or 'unpacking' the structural rules:

$$\dfrac{}{\vdash \Gamma, p, \overline{p}}\,\text{ax} \quad \sim \quad \dfrac{\dfrac{}{\vdash p, \overline{p}}\,\text{ax}}{\vdash \Gamma, p, \overline{p}}\,\text{W}$$

$$\dfrac{\vdash \Gamma, A \qquad \vdash \Gamma, B}{\vdash \Gamma, A \wedge B}\,\wedge \quad \sim \quad \dfrac{\dfrac{\vdash \Gamma, A \qquad \vdash \Gamma, B}{\vdash \Gamma, \Gamma, A \wedge B}\,\wedge}{\vdash \Gamma, A \wedge B}\,\text{C}$$

Observation

- *Logical rules are height-preserving invertible;*
- *Cut is admissible;*
- *The other structural rules are height-preserving admissible.*

Comparing LK and G4

Observation (Five Clues)

- *Semantic completeness;*
- *Algebraic nature of inference;*
- *Identity of proofs;*
- *Proof/Refutation;*
- *Cut elimination.*

Assumption ($Syntax \leftrightarrow Semantics$)

The more conceptually distant syntax and semantics are, the more laborious the completeness proof is expected to be.

Observation (Semantic Completeness for G4)

Semantic completeness for G4 *can be demonstrated without much effort:*

- *the rules of G4 preserve validity both upward and downward;*
- *and derivations reach conjunctive normal form upward:* $leaf_1 \wedge leaf_2 \wedge ... \wedge leaf_n$*, where* $leaf_i$ *is the* i*-est leaf, with form* $p, q, \overline{p}$ *and meaning* $p \vee q \vee ... \vee \overline{p}$*.*

If in the proof-search we reach a leaf that is not an axiom, we can construct a countermodel:

$$v(p) = \begin{cases} 0 & \textit{iff } p \in \textit{leaf} \\ 1 & \textit{iff } p \notin \textit{leaf} \end{cases}$$

Observation

- *Completeness for LK is usually proved indirectly, by proving derivable a complete list of axioms;*
- *The proof for G4 does not extend to LK:*
  - *Rules are not valid upward;*
  - *Contraction could be applied ad libitum upward;*
  - *Weakening applied upward could erase a formula needed to close the derivation.*

Observation (Completeness for LK)

A direct proof of completeness is significantly less simple than the one for G4.

Algebraic Nature of Inferences

Observation (Algebraic Reading of G4)

In G4, rules establish an equivalence between the (conjunction of the) premises and the conclusion.

$$\frac{\vdash \Gamma, A \wedge B}{\vdash \Gamma, A \qquad \vdash \Gamma, B} \qquad \prod \Gamma \cdot (A + B) = \prod \Gamma \cdot A + \prod \Gamma \cdot B$$

Example

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\vdash p, \overline{p}, q, r \qquad \vdash q, \overline{q}, \overline{p}, r}{\vdash p \wedge \overline{q}, \overline{p}, q, r}\wedge \qquad \vdash r, \overline{r}, \overline{p}, q}{\vdash (p \wedge \overline{q}) \wedge \overline{r}, \overline{p}, q, r}\wedge}{\vdash (p \wedge \overline{q}) \wedge \overline{r}, \overline{p}, q \vee r}\vee}{\vdash (p \wedge \overline{q}) \wedge \overline{r}, \overline{p} \vee (q \vee r)}\vee}{\vdash ((p \wedge \overline{q}) \wedge \overline{r}) \vee (\overline{p} \vee (q \vee r))}\vee$$

$$\dfrac{\dfrac{\dfrac{\dfrac{((p + \overline{q}) + \overline{r}) \cdot (\overline{p} \cdot (q \cdot r))}{((p + \overline{q}) + \overline{r}) \cdot \overline{p} \cdot (q \cdot r)}}{((p + \overline{q}) + \overline{r}) \cdot \overline{p} \cdot q \cdot r}}{[(p + \overline{q}) \cdot \overline{p} \cdot q \cdot r] + (r \cdot \overline{r} \cdot \overline{p} \cdot q)}}{[(p \cdot \overline{p} \cdot q \cdot r) + (q \cdot \overline{q} \cdot \overline{p} \cdot r)] + (r \cdot \overline{r} \cdot \overline{p} \cdot q)}$$

Observation (For LK it doesn't hold)

$$\frac{\vdash \Gamma, A \vee B}{\vdash \Gamma, A} \qquad \prod \Gamma \cdot (A \cdot B) \neq \prod \Gamma \cdot A$$

Identity of Proofs

Assumption (Identity of proofs up to permutation of rules)

Identity of derivations in sequent calculus should be evaluated up to permutations of rules:

$$\dfrac{\dfrac{\vdash p, \overline{p}}{\vdash p \vee q, \overline{p}}\ \vee}{\vdash p \vee q, \overline{p} \vee r}\ \vee \qquad \dfrac{\dfrac{\vdash p, \overline{p}}{\vdash p, \overline{p} \vee r}\ \vee}{\vdash p \vee q, \overline{p} \vee r}\ \vee$$

- *Rules in natural deduction cannot be rearranged in the same way;*
- *An intuition shared with combinatorial proofs.*

Observation (One Proof in G4)

- *In G4, it can be shown that there is exactly one Cut-free proof for each provable sequent, up to permutation of the inferences.*

Observation (Many Proofs in LK)

- *On the contrary, in* LK *provable sequents have more than one Cut-free proof.*

Example

Both these derivations are Cut-free, but the one on the left cannot be executed in G4:

$$\dfrac{\dfrac{\dfrac{\dfrac{\vdash p, \overline{p}}{\vdash p, \overline{p}, q}\ \mathsf{W}}{\vdash p \vee q, \overline{p}, q}\ \vee}{\vdash p \vee q, \overline{p}, p \vee q}\ \vee}{\vdash p \vee q, \overline{p}}\ \mathsf{C} \qquad\qquad \dfrac{\vdash p, \overline{p}}{\vdash p \vee q, \overline{p}}\ \vee$$

Observation (Semantic Pollution)

G4 cannot represent some informal reasoning:

> *We have two alternatives: p can be true or not. Let us consider moreover a third case in which q is true. In the first case, since p is true, then $p \vee q$ is true, and in the third case, since q is true, then $p \vee q$ is true. But these alternatives are identical, so we have two cases: in one p does not hold, in the other $p \vee q$ does.*

Proof/Refutation

Observation (In G4)

By applying the rules of G4 bottom-up, we reach:

- *a proof of the sequent; or*
- *a refutation of the sequent.*

Observation (In LK)

Proof-search can fail in LK also for provable sequents, because of:

- *the additive formulation of disjunction (we could select the wrong disjunct);*
- *the multiplicative formulation of conjunction (wrong splitting of the context);*
- *Weakening (erase the wrong sentence);*
- *Contraction (applicable ad libitum)*

A First Criterion for Semantic Pollution

Observation (Points in favor of LK)

The approach toward structural rules in G4 *forgoes the representation of demonstrative practices for practicality, bridging the gap with semantics:*

- *the additive formulation of disjunction seems more apt at characterising the meaning of disjunction, since its multiplicative formulation could be regarded as circular, the comma itself being interpreted as a disjunction;*
- *Weakening corresponds to the evaluation of an alternative situation in which the main sentence holds;*
- *Contraction to the recognition of two identical conclusions derived in different situations.*

Assumption (A first *criterion* for semantic pollution)

In general, a system is more polluted semantically if it gives up the possibility of representing some proof-techniques in order to simplify the proof of metatheoretical properties or the automatic construction of derivations. As for G4 the absorption of structural rules in the operational ones limits their applicability, preventing the evaluation of some proof strategies.

Cut Elimination

Observation (Cut elimination for G4)

- *To prove Cut-elimination in the usual way:*
  - *induction up to ω for Reduction Lemma;*
    - *for any application of Cut the derivations of its premises can be rearranged so to introduce the Cut-formula with their last inference;*
    - *each step of Cut elimination reduces the complexity of the Cut formula, and induction on this parameter is sufficient to prove the lemma.*
  - *induction up to $\omega^2 + \omega$ for Cut-elimination.*
    - *primary induction on the maximal complexity of a Cut-formula in the derivation;*
    - *the secondary one on the number of maximal applications of Cut.*
- *Cut-elimination can be proved also by unthreading, a procedure that clearly highlights the* ***redundancy of this rule****.*

Observation (Cut-free derivations)

Each step of Cut elimination reduces the length of the derivation, so that ***Cut-free derivations*** *are* ***shorter*** *than derivations with Cut.*

Assumption (Semantic pollution)

Cut must be eliminable, but it must also have sense to use it.

Observation (Cut elimination for LK)

- *Reduction Lemma is not proved directly for Cut, but for the rule of Mix;*
- *induction up to $\omega^2 + \omega$ for Reduction Lemma;*
- *induction up to $\omega^3 + \omega^2 + \omega$ for Cut-elimination.*
  - *primary induction is on the number of applications of Mix;*
  - *secondary induction is on the degree of the topmost Mix formula;*
  - *third induction is on the rank of the topmost Mix formula.*

Observation (Cut and length of derivations)

Cut elimination causes an increase in the length of the derivation, and so the rule of Cut, even though dispensable, is useful.

Observation (The role of Cut)

G4 fails to convey what is the role of Cut in demonstrative practice.

Assumption (The role of structural rules)

In G4 *Weakening and Contraction are height-preserving admissible;*

In LK *Weakening and Contraction for non-atomic sentences are admissible.*

LK capture the role of Weakening and Contraction, while G4 does not.

Conclusion

Conclusions

Observation (First Part)

- *As for explicit pollution:*
  - *We should clearly distinguish between proof-analysis and proof-theoretic justification;*
  - *It is not problematic for proof-analysis;*
    - *Negri's proof that there are no modal logic for irreflexive R;*
  - *It is controversial whether it is problematic for proof-theoretic justification (Read);*
    - *Negri and Read's systems have problems independent of semantic pollution;*
- *As for implicit pollution:*
  - *Pollution goes in both directions:*
    - *Realist interpretation of a system extends to each isomorphic system;*
    - *Inferentialist interpretation of a system extend to each isomorphic system.*
  - *We have two options:*
    - *Non-formal meaning-theoretical criteria for the acceptability of logical systems (Dummett);*
    - *Internal criteria independent of isomorphism with 'semantic' systems (second part).*

Observation (Second Part)

- *There are different degrees of semantic pollution;*
- *Absorbing the structural rules into the logical ones (and axiom) pollutes the system;*
- *This makes logical rules height-preserving invertible and structural rules height-preserving admissible;*
- *Hence, it makes easier the proof of some metateoretical results:*
  - *Cut Elimination;*
  - *Semantic Completeness;*
- *We can even algebraically read derivations;*
- *But, in order to do so:*
  - ***We need to restrict the class of constructible arguments:** in G4, only one proof up permutation of rules.*
  - ***Cut loses its role:** Cut-free derivations are shorter than non-normal derivations.*

Thanks for your attention!

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