Semantic Pollution for Modal and Classical Logic

Work in progress with Gabriele Pulcini

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Proof-Theoretic Criteria and Semantic Pollution

Assumption (An internal criterion for proof systems)

- *The **rules** should **characterize** the **logical terms**;*
- ***Nothing external** to the **content** of the theorem should be **used** in the proof (**Analytic** proofs).*

Observation (Bolzano)

***Bolzano**, in purely analytic proof of the theorem that between any two values which give results of opposite sign there lies at least one real root of the equation:*

> *... the proofs of the science should not merely be **certainty-makers**, but rather **grounding**, i.e. presentations of the **objective reason for** the **truth** concerned ...*

Assumption (Semantic Pollution)

- ***The proof-system should not incorporate extraneous semantic elements*** *(syntactic purity);*
  - *Semantic pollution makes unacceptable a proof-system that follows all the other criteria.*
- ***Proof procedures*** *should be syntactically pure;*
  - ***Semantic proof*** *of Cut-elimination is* ***not satisfactory****.*

Observation (Purity of Methods)

Internal criteria *supplemented with* ***syntactic purity*** *want to provide a formal interpretation of* ***purity of methods applied to inferential systems****.*

Criteria on rules and on proofs

Observation (Requirements for rules)

Separation: *The rules for each connective/modality should be* ***independent of any other connective/modality****;*

Weak Symmetry: *Each rule should* ***either be a left or a right rule****.*

Symmetry ***Both*** *left and right rules for each connective/modality.*

Observation (Requirements for proofs)

Cut Elimination: *Cut should be* ***eliminable****;*

Subformula Property: *For every provable sequent, there is a proof in which only* ***subformulas*** *of the* ***formulas*** *in the* ***endsequent*** *occur.*

From Explicit to Implicit Pollution

Modal logic G3K

$$\Gamma, wRo \Rightarrow wRo, \Delta \qquad\qquad \Gamma, w : p \Rightarrow w : p, \Delta$$

$$\Box \Rightarrow \frac{\Gamma, o : A, w : \Box A, wRo \Rightarrow \Delta}{\Gamma, w : \Box A, wRo \Rightarrow \Delta} \qquad \Rightarrow \Box \frac{\Gamma, wRo \Rightarrow o : A, \Delta}{\Gamma \Rightarrow w : \Box A, \Delta}$$

$$\Diamond \Rightarrow \frac{\Gamma, o : A, wRo \Rightarrow \Delta}{\Gamma, w : \Diamond A \Rightarrow \Delta} \qquad \Rightarrow \Diamond \frac{\Gamma, wRo \Rightarrow w : \Diamond A, o : A, \Delta}{\Gamma, wRo \Rightarrow w : \Diamond A, \Delta}$$

$$Weak \Rightarrow \frac{\Gamma \Rightarrow \Delta}{\Gamma, w : A \Rightarrow \Delta} \qquad \Rightarrow Weak \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow w : A, \Delta}$$

$$Weak \Rightarrow \frac{\Gamma \Rightarrow \Delta}{\Gamma, wRo \Rightarrow \Delta} \qquad \Rightarrow Weak \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow wRo, \Delta}$$

$$Cut \frac{\Gamma \Rightarrow w : A, \Delta \qquad \Theta, w : A \Rightarrow \Lambda}{\Gamma, \Theta \Rightarrow \Delta, \Lambda}$$

$$\wedge \Rightarrow \frac{\Gamma, w : A, , w : B \Rightarrow \Delta}{\Gamma, w : A \wedge B \Rightarrow \Delta} \qquad \wedge \Rightarrow \frac{\Gamma, w : B \Rightarrow \Delta}{\Gamma, w : A \wedge B \Rightarrow \Delta}$$

Observation

In $\Box R$ and $\Diamond L$ o is a new label.

Example (Basic Rules)

$$\Gamma, wRo \Rightarrow wRo, \Delta \qquad \Gamma, w : p \Rightarrow w : p, \Delta \qquad \Rightarrow\Box\ \frac{\Gamma, wRo \Rightarrow o : A, \Delta}{\Gamma \Rightarrow w : \Box A, \Delta}$$

With o a new label.

Example (Rules for accessibility)

$$\frac{wRw, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\ \text{Ref} \qquad \frac{wRr, wRo, oRr, \Gamma \Rightarrow \Delta}{wRo, oRr, \Gamma \Rightarrow \Delta}\ \text{Trans} \qquad \frac{oRw, wRo, \Gamma \Rightarrow \Delta}{wRo, \Gamma \Rightarrow \Delta}\ \text{Sym}$$

Observation

G3T = *G3K* + *Ref*

G3S4 = *G3K* + *Ref* + *Trans*

G3S5 = *G3K* + *Ref* + *Trans* + *Sym*

(Viganó 2000) (Negri 2005)

Points in favor

Observation (Good Properties)

- *Cut elimination;*
- *Height-preserving admissibility of structural rules;*
- *Height-preserving invertibility of operational rules;*
- *Axiom is already atomic.*
- *These good properties are preserved in the extensions.*

Observation (Došen's Principle and R)

Negri argues that the system suit Došen's Principle, since the different modal systems can be obtained by changing 'structural' rules for the accessibility relation.

Objection (Problems)

Negri's system has problems that are independent of semantic pollution, but let us ignore this issue for the moment.

Skip

Semantic pollution!

Objection (Elements from Kripke's Frames)

Negri's modal systems is semantically polluted since elements of Kripke's structure occur in it.

Objection (What's wrong with Semantic Pollution)

Semantically polluted systems should be avoided because:

- *The modalities are not characterized proof-theoretically, but by reference to the clauses for Kripke's frames.*

Objection (Irreflexive R)

Negri's system can describe accessibility relations that does not correspond to any modal logic, like irreflexive accessibility relations.

Answer (Good Proof Analysis)

*The extension of **G3K** with the rule* $\dfrac{}{wRw, \Gamma \Rightarrow \Delta}$ *Irreflex* *is conservative.*

> *Suppose that the sequent $\Gamma \Rightarrow \Delta$ (not containing uRv) is derivable in **G3K** + **Irref**. The atoms of the form xRy that appear on the left-hand side of sequents in the derivation originate from applications of rule $\Rightarrow \Box$. By the variable condition, $x \neq y$, so the derivation contains no atom of the form xRx, hence no application of **Irref**. Therefore the sequent is derivable in **G3K**.*

Theorem 7.1 of Negri, Proof Analysis in Modal Logic, 2005.

Observation (Good proof-analysis)

Negri's system is good for proof analysis, and the procedures of proof are not semantically polluted. The system itself seems polluted instead.

Skip

Objection (No-Symmetry (I or E-rules))

Natural deduction rules are simultaneously (general)I and (general)E-rules:

$$\dfrac{\begin{array}{c}[wRw]\\ \vdots\\ u:A\end{array}}{u:A}\;\textit{T (Refl)} \qquad \dfrac{wRv \qquad vRu \qquad \begin{array}{c}[wRu]\\ \vdots\\ t:A\end{array}}{t:A}\;\textit{4 (Trans)} \qquad \dfrac{wRv \qquad \begin{array}{c}[vRw]\\ \vdots\\ t:A\end{array}}{t:A}\;\textit{B (Sym)}$$

Even more obvious with the rules:

$$\dfrac{}{wRw}\;\textit{T (Refl)} \qquad \dfrac{wRv \qquad vRu}{wRu}\;\textit{4 (Trans)} \qquad \dfrac{wRv}{vRw}\;\textit{B (Symm)}$$

Objection (Harmony?)

The maximal formula

$$\dfrac{\dfrac{uRw}{wRu}\,B \qquad uRv}{wRv}\,4$$

cannot be obtained only because of the restriction on the minor premises of Read's rules.

- *Read interprets harmony as functionality of E-rules upon I-rules!*

Objection (No-Symmetry (Left or Right-rules))

Negri's rules are in some sense Left-rules, since operate with left hand side of $\Rightarrow$. However, they remove sentences instead of introducing them.

$$\frac{wRw, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\,\textit{Ref} \qquad \frac{wRr, wRo, oRr, \Gamma \Rightarrow \Delta}{wRo, oRr, \Gamma \Rightarrow \Delta}\,\textit{Trans}$$

$$\frac{oRw, wRo, \Gamma \Rightarrow \Delta}{wRo, \Gamma \Rightarrow \Delta}\,\textit{Sym} \qquad \frac{wRr, wRo, oRr, \Gamma \Rightarrow \Delta}{wRo, wRr, \Gamma \Rightarrow \Delta}\,\textit{Eucl}$$

Objection (Cut-elimination?)

Using groundsequents for the assumptions about accessibility relations, we end up with Cuts, while using Negri's rules these can be avoided:

$$\frac{\dfrac{}{uRw \Rightarrow wRu}\,\textit{Sym} \qquad \dfrac{}{wRu, uRv \Rightarrow wRv}\,\textit{Trans}}{uRw, uRv \Rightarrow wRv}\,\textit{Cut} \quad \sim \quad \dfrac{\dfrac{\dfrac{}{uRw, wRu, uRv, wRv \Rightarrow wRv}\,\textit{Axiom}}{uRw, wRu, uRv \Rightarrow wRv}\,\textit{Trans}}{uRw, uRv \Rightarrow wRv}\,\textit{Sym}$$

Negri's rules delete formulas from the premises because they absorb Cut:

$$\frac{\dfrac{}{wRo \Rightarrow oRw}\,\textit{Sym} \qquad wRo, oRw, \Gamma \Rightarrow \Delta}{wRo, \Gamma \Rightarrow \Delta}\,\textit{Cut} \quad \sim \quad \frac{wRo, oRw, \Gamma \Rightarrow \Delta}{wRo, \Gamma \Rightarrow \Delta}\,\textit{Sym}$$

Skip

Objection (Contraction is not really admissible)

When Ref is not present, we have to assume a contracted version of the $R-rules$ with more than one formula for accessibility in the conclusion.

$$\frac{wRw, wRw, wRw, \Gamma \Rightarrow \Delta}{wRw, wRw, \Gamma \Rightarrow \Delta}\,\textit{Trans} \qquad \frac{wRw, wRw, \Gamma \Rightarrow \Delta}{wRw, \Gamma \Rightarrow \Delta}\,\textit{Trans*}$$

Answer (Proof-search (Negri) and atomic contraction)

- *Contracted rules do not cause troubles for proof-search, since there is a bounded number of possible cases of contracted rules to be added.*
- *The only ineliminable contractions regard relational atoms, so the meaning of logical terms is independent of Contraction (Hacking).*

Example (Non-atomic weakening for S4)

In Ohinshi and Matsumoto's system

$$\frac{\Gamma, A \Rightarrow \Delta}{\Gamma, \Box A \Rightarrow \Delta}\,\mathrm{L}\Box \qquad \frac{\Box\Gamma \Rightarrow A}{\Box\Gamma \Rightarrow \Box A}\,\mathrm{R}\Box$$

weakening cannot be reduced to its atomic form: consider $\Box A \Rightarrow \Box A, \Box B$.

Objection (Lack of subformula property)

Negri's systems do not satisfy the subformula property, relational atoms occur in a Cut-free derivation together with the subformulas of the formulas in the endsequent.

Answer (Subterm property)

All terms in minimal derivations are terms found in the endsequent.

Observation (Proof-Analysis and PT-justification)

- *This is enough for proof-search and decidability (Negri).*
- *Is it enough for **analytic** proofs?*

Problems

Objection (Problems)

- *Subformula property doesn't hold;*
- *Closure Condition asks for the extension with extra rules when Refl is absent (to have Contraction admissible);*
- *Rules for $\Box$ and $\Diamond$ are separable, but both depend on R-rules;*
- *Symmetry and Explicitness hold for $\Box$ and $\Diamond$-rules but not for R-rules:*
  - *The rules are not properly L or R-rules;*
  - *R occur both in the conclusion and in the premises;*
- *R-rules are structural? (Does Došen principle hold?)*
- ***The calculus is semantically polluted by the reference to Kripke's frames.***

Syntactically Pure Systems

Example (Tree-hypersequents)

Poggiolesi proposes a system that **preserves** the **tree-structure** of Kripke's frames but does **not** use explicit **labels**:

$$\frac{G\{\Box A, M \Rightarrow N, [A, P \Rightarrow Q]\}}{G\{\Box A, M \Rightarrow N, [P \Rightarrow Q]\}}\,\Box\mathrm{L} \quad \sim \quad \frac{\Gamma, w : \Box A, wRu \Rightarrow u : A, \Delta}{\Gamma, w : \Box A \Rightarrow \Delta}\,\Box\mathrm{L}$$

$$\frac{G\{M \Rightarrow N, [\Rightarrow A]\}}{G\{M \Rightarrow N, \Box A\}}\,\Box\mathrm{R} \quad \sim \quad \frac{\Gamma, wRo \Rightarrow o : A, \Delta}{\Gamma \Rightarrow w : \Box A, \Delta}\,\Box\mathrm{R}$$

Nested Sequents (Bull & Kashima), Deep Sequents (Brünnler)

Observation (Subformula and Contraction)

- *Poggiolesi's systems satisfy the **subformula property**;*
- ***Contraction** is **admissible** in Poggiolesi's system, without need for the Closure Condition;*
- *It is syntactically pure (**no labels**).*

Notational Variation

Observation (Translations between systems)

Labeled sequents are a generalization of tree-hypersequents:

- *The **accessibility** relation is a **treelike** relation;*
- ***Every label** occurring in a formula also occurs in a relational **atom**.*

(Goré & Ramanayake 2012)

Question

*Is it **enough** to gain **purity**?*

Observation (Topical Purity)

Detlefsen and Arana (2011):

> *We say that a solution ε of P is **topically pure** when it draws only on such commitments as topically determine P.*

Commitments $=$ *definitions, axioms, inferences, etc.*

Topically determining commitments $=$ *commitments that determine the content of the problem.*

Observation (Ontological Purity)

Martinot (2024) proposes:

- *a criterion of **full** ontological purity which generalizes topical purity with definitional extensions;*
- *a criterion of **secondary** ontological purity which generalizes **full** ontological purity with proofs that are **translatable** into a pure proof.*

Implicit Semantic Pollution

Observation (Implicit Pollution)

According to Avron, a calculus can be semantically polluted even if no semantical label occur in it:

> *Because of the proof-theoretical nature and the expected generality, the framework should be independent of any particular semantics.* ***One should not be able to guess, just from the form of the structures which are used, the intended semantics of a given proof system.***

Avron, The method of hypersequents, 1996.

Objection (Poggiolesi's systems are (implicitly) semantically polluted)

- *They can be* ***translated*** *into* ***Negri****'s systems;*
- *They are* ***inspired*** *by Kripke's* ***frames****.*

Answer (Poggiolesi)

- ***Even*** *some* ***sequents*** *for* ***classical logic*** *are (implicitly) semantically* ***polluted****;*

Objection (Poggiolesi-Read)

Kleene's context-sharing classical ***sequent*** *calculi are* ***isomorphic*** *to* ***tableaux-systems****, which are regarded as clearly semantically inspired (even '****hybrids****' between* ***syntactic*** *and* ***semantic*** *systems).*

$$\dfrac{\dfrac{\dfrac{p \Rightarrow p, \neg q}{\Rightarrow p, \neg p, \neg q} \qquad \dfrac{q \Rightarrow q, \neg p}{\Rightarrow q, \neg p, \neg q}}{\dfrac{\Rightarrow p \wedge q, \neg p, \neg q}{\Rightarrow p \wedge q, \neg p \vee \neg q}}}{\neg(p \wedge q) \Rightarrow \neg p \vee \neg q}$$

$$\begin{array}{c} T\neg(p \wedge q) \\ F\neg p \vee \neg q \\ Fp \wedge q \\ F\neg p \\ F\neg q \\ Fp \mid Fq \\ Fq \mid Tq \\ \times \quad \times \end{array}$$

So, even sequent systems for classical logic should count as semantically polluted. Hence, ***implicit semantic pollution is too broadly applicable to be a good criterion****.*

Observation (Inferential Reading of Classical Logic)

- ***Bilateral*** *(assertion/rejection)* ***reading*** *of tableaux-systems (Priest) and sequent calculus (Read);*
- *In tableaux-systems,* ***T*** *can be omitted and* ***F*** *can be converted in* $\neg$.

Observation (Inferential Interpretation of Hypersequents)

Poggiolesi and Restall propose an ***inferential interpretation*** *of tree-hypersequents:*

> *Suppose* $\Diamond(A \vee B)$. *So,* ***in some circumstance***, $A \vee B$. *There are two cases: Case (i) A, and Case (ii) B. Take case (i) first. Then in this circumstance, A and so,* ***back where we started***, $\Diamond A$, *and hence* $\Diamond A \vee \Diamond B$. *On the other hand, we might have case (ii). There we have B, so back* ***in the original circumstance***, $\Diamond B$, *and hence,* $\Diamond A \vee \Diamond B$. *So, in either case, we have* $\Diamond A \vee \Diamond B$, *which is what we wanted.*

The idea is to interpret modal logic as speaking of ***different circumstances****, delimited by* ***slashes*** *in tree-hypersequents.*

Objection (Same Interpretation for Negri's system?)

The ***same interpretation*** *could be used for (properly restricted)* ***Negri****'s system!*

Objection (Two Directions of Pollution)

Pollution *goes in* ***both directions****:*

- *an* ***inferential*** *reading of a system can be extended to any* ***isomorphic*** *system;*
- *a* ***semantic*** *(model-theoretic) reading of a system can be extended to any* ***isomorphic*** *system.*

Observation (Structural Criteria?)

- *An* ***internal criterion*** *should* ***not*** *focus on* ***semantic interpretations*** *or* ***inspirations*** *of the proof-system,* ***but*** *on* ***structural results****.*

Degrees of Semantic Pollution for Classical Logic

LK	**G4**
$\dfrac{}{\vdash\ p, \overline{p}}$ ax	$\dfrac{}{\vdash\ \Gamma, p, \overline{p}}$ ax
$\dfrac{\vdash\ \Gamma, A \qquad \vdash\ \Delta, \overline{A}}{\vdash\ \Gamma, \Delta}$ Cut	$\dfrac{\vdash\ \Gamma, A \qquad \vdash\ \Gamma, \overline{A}}{\vdash\ \Gamma}$ Cut
$\dfrac{\vdash\ \Gamma, A}{\vdash\ \Gamma, A \vee B}\vee \qquad \dfrac{\vdash\ \Gamma, B}{\vdash\ \Gamma, A \vee B}\vee$	$\dfrac{\vdash\ \Gamma, A, B}{\vdash\ \Gamma, A \vee B}\vee$
$\dfrac{\vdash\ \Gamma, A \qquad \vdash\ \Delta, B}{\vdash\ \Gamma, A \wedge B, \Delta}\wedge$	$\dfrac{\vdash\ \Gamma, A \qquad \vdash\ \Gamma, B}{\vdash\ \Gamma, A \wedge B}\wedge$
$\dfrac{\vdash\ \Gamma}{\vdash\ \Gamma, A}$ W	$\dfrac{\vdash\ \Gamma, A, A}{\vdash\ \Gamma, A}$ C

Assumption (Fully atomic axioms)

All the formulas that occur in the axiom must be atomic.

From LK to G4

Observation ('Packing' or 'unpacking' structural rules)

Converting *derivations from* ***LK*** *to* ***G4*** *and vice-versa is simply a matter of '****packing****' or '****unpacking****' the* ***structural*** *rules:*

$$\dfrac{}{\vdash \Gamma, p, \overline{p}}\,\mathsf{ax} \quad \sim \quad \dfrac{\dfrac{}{\vdash p, \overline{p}}\,\mathsf{ax}}{\vdash \Gamma, p, \overline{p}}\,\mathsf{W}$$

$$\dfrac{\vdash \Gamma, A \qquad \vdash \Gamma, B}{\vdash \Gamma, A \wedge B}\,\wedge \quad \sim \quad \dfrac{\dfrac{\vdash \Gamma, A \qquad \vdash \Gamma, B}{\vdash \Gamma, \Gamma, A \wedge B}\,\wedge}{\vdash \Gamma, A \wedge B}\,\mathsf{C}$$

Observation

- ***Logical*** *rules are* ***height-preserving invertible****;*
- ***Cut*** *is* ***admissible****;*
- *The other* ***structural*** *rules are* ***height-preserving admissible****.*

Comparing LK and G4

Assumption (Syntax-Semantic)

*There are **five** formal **results** that characterize the **relation** between **syntax** and **semantics**.*

Observation (Five Clues)

- *Semantic completeness;*
- *Algebraic nature of inference;*
- *Identity of proofs;*
- *Proof/Refutation;*
- *Cut elimination.*

Semantic Completeness

Assumption ($Syntax \leftrightarrow Semantics$)

*The **more** conceptually **distant syntax** and **semantics** are, the **more laborious** the completeness **proof** is expected to be.*

Observation (Semantic Completeness for G4)

Semantic completeness for G4 *can be demonstrated **without** much **effort**:*

- *the rules of G4 preserve **validity** both **upward** and **downward**;*
- *derivations **reach conjunctive normal form upward**:* $leaf_1 \wedge leaf_2 \wedge ... \wedge leaf_n$, *where* $leaf_i$ *is the* i*-est leaf, with form* $p, q, \overline{p}$ *and meaning* $p \vee q \vee ... \vee \overline{p}$.

*If in the proof-search we reach a leaf that is **not** an **axiom**, we can construct a **countermodel**:*

$$v(p) = \begin{cases} 0 & \textit{iff } p \in \textit{leaf} \\ 1 & \textit{iff } p \notin \textit{leaf} \end{cases}$$

Observation

- *Completeness for **LK** is usually proved **indirectly**, by proving derivable a complete list of axioms;*
- *The proof for G4 does not extend to LK:*
  - *Rules are not valid upward;*
  - *Contraction could be applied ad libitum upward;*
  - *Weakening applied upward could erase a formula needed to close the derivation.*

Observation (Completeness for LK)

*A **direct proof** of completeness is significantly **less simple** than the one for G4.*

Algebraic Nature of Inferences

Observation (Algebraic Reading of G4)

*In G4, rules establish an **equivalence between** the (conjunction of the) **premises** and the **conclusion**.*

$$\frac{\vdash \Gamma, A \wedge B}{\vdash \Gamma, A \quad \vdash \Gamma, B} \qquad \prod \Gamma \cdot (A + B) = \prod \Gamma \cdot A + \prod \Gamma \cdot B$$

Example

$$\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\vdash p, \overline{p}, q, r \qquad \vdash q, \overline{q}, \overline{p}, r}{\vdash p \wedge \overline{q}, \overline{p}, q, r}\wedge \qquad \vdash r, \overline{r}, \overline{p}, q}{\vdash (p \wedge \overline{q}) \wedge \overline{r}, \overline{p}, q, r}\wedge}{\vdash (p \wedge \overline{q}) \wedge \overline{r}, \overline{p}, q \vee r}\vee}{\vdash (p \wedge \overline{q}) \wedge \overline{r}, \overline{p} \vee (q \vee r)}\vee}{\vdash ((p \wedge \overline{q}) \wedge \overline{r}) \vee (\overline{p} \vee (q \vee r))}\vee$$

$$\cfrac{\cfrac{\cfrac{\cfrac{((p + \overline{q}) + \overline{r}) \cdot (\overline{p} \cdot (q \cdot r))}{((p + \overline{q}) + \overline{r}) \cdot \overline{p} \cdot (q \cdot r)}}{((p + \overline{q}) + \overline{r}) \cdot \overline{p} \cdot q \cdot r}}{[(p + \overline{q}) \cdot \overline{p} \cdot q \cdot r] + (r \cdot \overline{r} \cdot \overline{p} \cdot q)}}{[(p \cdot \overline{p} \cdot q \cdot r) + (q \cdot \overline{q} \cdot \overline{p} \cdot r)] + (r \cdot \overline{r} \cdot \overline{p} \cdot q)}$$

Observation (For LK it doesn't hold)

$$\frac{\vdash \Gamma, A \vee B}{\vdash \Gamma, A} \qquad \prod \Gamma \cdot (A \cdot B) \neq \prod \Gamma \cdot A$$

Identity of Proofs

Assumption (Identity of proofs up to permutation of rules)

Identity *of derivations in sequent calculus should be evaluated* ***up to permutations*** *of* ***rules****:*

$$\dfrac{\dfrac{\vdash p, \overline{p}}{\vdash p \vee q, \overline{p}}\vee}{\vdash p \vee q, \overline{p} \vee r}\vee \qquad \dfrac{\dfrac{\vdash p, \overline{p}}{\vdash p, \overline{p} \vee r}\vee}{\vdash p \vee q, \overline{p} \vee r}\vee$$

- *Rules in* ***natural deduction cannot*** *be* ***rearranged*** *in the same way;*
- *An intuition shared with* ***combinatorial proofs****.*

Observation (One Proof in G4)

- *In G4, it can be shown that there is* ***exactly one Cut-free proof*** *for each provable sequent, up to permutation of the inferences.*

Observation (Many Proofs in LK)

- *On the contrary, in* LK *provable sequents have* ***more than one Cut-free proof***.

Example

Both these derivations are Cut-free, but the one on the left cannot be executed in G4:

$$\dfrac{\dfrac{\dfrac{\dfrac{\vdash p, \overline{p}}{\vdash p, \overline{p}, q}\,\mathsf{W}}{\vdash p \vee q, \overline{p}, q}\,\vee}{\vdash p \vee q, \overline{p}, p \vee q}\,\vee}{\vdash p \vee q, \overline{p}}\,\mathsf{C} \qquad\qquad \dfrac{\vdash p, \overline{p}}{\vdash p \vee q, \overline{p}}\,\vee$$

Observation (Semantic Pollution)

G4 ***cannot represent*** *some* ***informal reasoning****:*

> *We have two* ***alternatives****: p can be true or not. Let us* ***consider*** *moreover a* ***third case*** *in which q is true. In the first case, since p is true, then $p \vee q$ is true, and in the third case, since q is true, then $p \vee q$ is true. But* ***these alternatives are identical****, so we have two cases: in one p does not hold, in the other $p \vee q$ does.*

Observation (In G4)

*By applying the **rules** of G4 **bottom-up**, we reach:*

- *a **proof** of the sequent; or*
- *a **refutation** of the sequent.*

Observation (In LK)

*Proof-**search can fail** in LK also for **provable** sequents, because of:*

- *the **additive** formulation of **disjunction** (we could select the wrong disjunct);*
- *the **multiplicative** formulation of **conjunction** (wrong splitting of the context);*
- ***Weakening** (erase the wrong sentence);*
- ***Contraction** (applicable **ad libitum**)*

Points in Favor of LK

Observation (Points in favor of LK)

G4 ***forgoes*** *the representation of informal* ***proofs*** *for practicality,* ***bridging*** *the* ***gap*** *with* ***semantics****:*

- *the* ***additive*** *formulation of* ***disjunction*** *seems more apt at characterising the meaning of disjunction, since its* ***multiplicative*** *formulation could be regarded as* ***circular*** *(comma = disjunction)*

$$\frac{\vdash \ \Gamma, A}{\vdash \ \Gamma, A \vee B} \vee \qquad \frac{\vdash \ \Gamma, A, B}{\vdash \ \Gamma, A \vee B} \vee$$

- ***Weakening*** *corresponds to the* ***evaluation*** *of an* ***alternative situation*** *in which the main sentence holds: it* ***should not be restricted****.*

- ***Contraction*** *to the* ***recognition*** *of two* ***identical*** *conclusions derived in different situations: it* ***should not be restricted****.*

Assumption (A first *criterion* for semantic pollution)

*A system is **more polluted** semantically **if***

- *it **drops** some **proof-techniques** in order to **simplify** the **proof** of **metatheoretical** properties or the **automatic** construction of derivations;*
- *this is true especially for those **metatheoretical** results that connect **syntax** with **semantics**.*

Cut Elimination

Observation (Cut elimination for G4)

- *To prove Cut-elimination in the usual way:*
  - ***induction*** *up to* ω *for* ***Reduction Lemma****;*
    - *for any application of Cut the derivations of its premises can be* ***rearranged*** *so to introduce the* ***Cut-formula*** *with their* ***last inference****;*
    - *each step of Cut elimination* ***reduces*** *the* ***complexity*** *of the* ***Cut-formula****, and induction on this parameter is sufficient to prove the lemma.*
  - ***induction*** *up to* $\omega^2 + \omega$ *for* ***Cut-elimination****.*
    - ***primary*** *induction on the* ***maximal complexity*** *of a Cut-formula in the derivation;*
    - *the* ***secondary*** *one on the* ***number of maximal*** *applications of Cut.*
- *Cut-elimination can be proved also by* ***unthreading****, a procedure that clearly* ***highlights*** *the* ***redundancy of this rule****.*

Observation (Cut-free derivations)

Each step of Cut elimination reduces the length of the derivation, so that ***Cut-free derivations*** *are* ***shorter*** *than derivations with Cut.*

Assumption (The role of Cut)

Cut must be ***eliminable****, but it must also* ***make sense*** *to use it.*

Observation (Cut elimination for LK)

- ***Reduction Lemma*** *is not proved directly for Cut, but for the rule of* ***Mix****;*
- *induction up to* $\omega^2 + \omega$ *for Reduction Lemma;*
- *induction up to* $\omega^3 + \omega^2 + \omega$ *for Cut-elimination.*
  - ***primary*** *induction is on the* ***number*** *of applications of Mix;*
  - ***secondary*** *induction is on the* ***degree*** *of the topmost Mix formula;*
  - ***third*** *induction is on the* ***rank*** *of the* ***topmost*** *Mix formula.*

Observation (Cut and length of derivations)

Cut elimination *causes an* ***increase*** *in the* ***length*** *of the derivation, and so the rule of* ***Cut****, even though dispensable, is* ***useful****.*

Observation (The role of Cut)

G4 fails to convey what is the role of Cut in demonstrative practice.

A general problem for structural rules?

Assumption (The role of structural rules)

In G4 ***Weakening*** *and* ***Contraction*** *are* ***height-preserving*** *admissible;*

In LK ***Weakening*** *and* ***Contraction for non-atomic sentences*** *are admissible.*

LK capture the role of Weakening and Contraction, while G4 does not. *A problem in general for logical variant of calculi (as opposed to general variant)?*

Conclusion

Observation

- *There are different* ***degrees*** *of* ***semantic pollution****;*
- ***Absorbing*** *the* ***structural rules*** *into the logical ones (and axiom)* ***pollutes*** *the system;*
- *This makes logical rules height-preserving* ***invertible*** *and structural rules* ***height-preserving admissible****;*
- *Hence, it makes* ***easier*** *the proof of some* ***metateoretical*** *results:*
  - ***Cut Elimination****;*
  - ***Semantic Completeness****;*
- *We can even algebraically read derivations;*
- *But, in order to do so:*
  - ***We need to restrict the class of constructible arguments:*** *in G4, only one proof up permutation of rules.*
  - ***Cut loses its role: Cut-free*** *derivations are* ***shorter*** *than non-normal derivations.*

Loose Ends

Observation

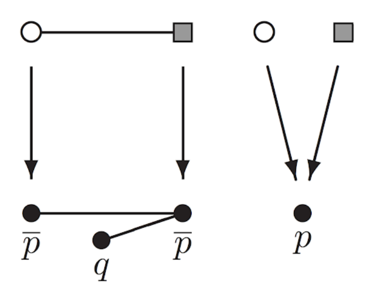
Brünnler, Deep sequent systems for modal logic (2009):

> *So they are **further removed from semantics**, more "syntactic" than labelled sequents. This shows in our **completeness proof**: we had to establish certain properties of, say, the euclidean closure of a relation, which is not needed for labelled systems. There, that relation is part of the system and it is being closed under euclideanness by the appropriate rule. It also shows in our **cut elimination procedure**: we had to show admissibility of certain rules in order to push the cut over the rules for the frame properties. This, again, is not needed for labelled systems. There the rules for the frame conditions do not affect the cut elimination procedure at all. So, in some sense we had to do more work in proving our systems complete.*

Assumption (Proofs without syntax)

By removing syntactic "noise", we obtain a criterion of identity for proofs.

Example (Peirce)



Objection (Semantic pollution?)

Can we do it without semantically polluting the system?

Thanks for your attention!

Bibliography

- Avron, A. (1996). The method of hypersequents in the proof theory of propositional non-classical logics. In Hodges, W., editor, Logic: Foundations to Applications, pages 1–32.
- Michael Detlefsen and Andrew Arana. Purity of methods. Philosopher's Imprint, 11(2), 2011.
- Dummett, M. (1991). The Logical Basis of Metaphysics. Harvard University Press, Cambridge (Massachussets).
- Dummett, M. (1996). Could there be unicorns? In The Seas of Language. Oxford University Press.
- Hacking, I. (1994). What is logic? In Gabbay, D., editor, What is a Logical System? Claredon Press, Oxford.
- Lellmann, B. and Poggiolesi, F. (2023). Saul Kripke on Modal logic, chapter Nested sequent or Tree-hypersequents: A survey. Springer, London.
- Negri, S. (2005). Proof analysis in modal logic. Journal of Philosophical Logic, 34:507–544.
- Negri, S. and von Plato, J. (2011). Proof Analysis: A Contribution to Hilbert's Last Problem. Cambridge University Press.

- Poggiolesi, F. (2010). Gentzen Calculi for Modal Propositional Logic. Springer Dordrecht.
- Poggiolesi, F. and Restall, G. (2012). New Waves in Philosophical Logic, chapter Interpreting and Applying Proof Theories for Modal Logic, pages 39–62. Palgrave Macmillan UK, London.
- Gabriele Pulcini. A note on cut-elimination for classical propositional logic. Archive for Mathematical Logic, pages 1–11, 2021.
- Gabriele Pulcini. Cut elimination by unthreading. Archive for Mathematical Logic, 63(1):211–223, 2023.
- Read, S. (2015). Semantic pollution and syntactic purity. Review of Symbolic Logic, 8:649–661.
- Greg Restall. Multiple conclusions. In Petr Hájek, Luis Valdés-Villanueva, and Dag Westersthl, editors, Logic, Methodology and Philosophy of Science. College Publications, 2005.
- Weiss, B. and Kürbis, N. (2024). Molecularity in the theory of meaning and the topic neutrality of logic. In D'Aragona, A. P., editor, Perspectives on Deduction: Contemporary Studies in the Philosophy, History and Formal Theories of Deduction, pages 187–209. Springer Verlag.