WHICH ECUMENICAL SYSTEM FOR CLASSICAL AND INTUITIONISTIC LOGICS?

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Abstract: Taking for granted that logic is grounded on the meaning of the logical terms, this paper deals with the issue of which logical constants are responsible for the disagreement between classical and intuitionistic logics. Dummett has argued that, since these logics are based on diverging theories of meaning, all the logical constants have different meanings. An ecumenical system (based on the negative translation) has been proposed by Prawitz as evidence that the disagreement between classical and intuitionistic logics is to some extent trivial, since there is a theory of meaning that justifies both of them, and based on a difference in the meaning of disjunction and conditional. In this paper, a different ecumenical system (based on the modal translation of intuitionistic logic into S4) is devised, and some good proof-theoretic properties are proved for it. It is shown that according to this system the disagreement between these logics is grounded on a difference in the meaning of negation and conditional, but not disjunction. Moreover, other differences between this and Prawitz's system are highlighted, and the consequences of the availability of both these systems on the debate regarding logical disagreement and the meaning of logical terms is discussed.

Keywords: Ecumenical Systems, intuitionistic logic, classical logic, modal logic, logical disagreement.

0 *Introduction: which meaning?*

Investigating the possibility of revising classical logic with quantum logic, Putnam claims that

> Only if it can be made out that [distributivity] is 'part of the meaning' of 'or' and/or 'and' (*which? and how does one decide?*) can it be maintained that quantum mechanics involves a 'change in the meaning' of one or both of these connectives. (Putnam, 1969, 233, italics mine)

Taking for granted that a positive solution can be provided for Putnam's main question 'are logical properties part of the meaning of the connectives?', we will focus on the secondary issue of providing a *criterion* to decide which constants are responsible for logical deviance. As a case study, in this paper we will deal with the disagreement between intuitionistic logic and classical logic. Hence, one of the questions we want to settle is whether the disagreement between classicists and intuitionists about the validity of excluded middle is based on:

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- A difference in the meaning attached to 'or';
- A difference in the meaning attached to 'not';
- A difference in the meaning attached to both 'or' and to 'not';

We will try to address this issue, explaining which formal and philosophical reasons there are to endorse one of these alternatives. We will not decide for one of the alternatives, but we will highlight the philosophical motivations that support each of them. In particular, we will see that the last alternative is supported by Dummett's perspective on the disagreement between classicists and intuitionists, while the alternative between the first two possibilities has been heavily neglected.

The structure of the paper is the following. Section 1 presents Dummett's position that logical disagreement between classicists and intuitionists should be considered as a deep disagreement. Section 2 presents Prawitz's objection that this disagreement can be conceived as trivial, since an ecumenical system that contains both classical and intuitionistic terms can be developed. Moreover, it is stressed that in this ecumenical system, the disagreement about excluded middle is based on a disagreement about the meaning of disjunction. Section 3 presents the modal translation of **I** into **S4**, which is used in section 4 to find an alternative ecumenical system for classical and intuitionistic logic. In this section, it is shown that this new ecumenical system ascribes the disagreement about excluded middle to a difference in the meaning of negation. Moreover, other differences are presented about the two ecumenical systems, and especially about how they relate classical and intuitionistic implications. Section 5 concludes.[1]

1 *Dummett's deep disagreement*

There is a well-known problem in developing a proof system that includes the rules for both classical and intuitionistic logic. Indeed, if no precaution is applied, the purely classical theorems become provable for the intuitionistic terms as well.[2] The easiest way to show this is to prove the interderivability of $\neg_i A$ and $\neg_c A$ in a system in which the standard rules for *intuitionistic* negation in natural deduction are derivable for both $\neg_i$ and $\neg_c$:

[1] We will use **I** to refer to intuitionistic propositional logic, **C** to refer to classical propositional logic and **S4** to refer to the $\Diamond$-free fragment of classical **S4** propositional logic. We will call ***I*** *wff* the well-formed-formulas for intuitionistic propositional logic defined as usual, ***C*** *wff* the well-formed-formulas for classical propositional logic defined as usual, and ***S4*** *wff* the well-formed-formulas for **S4** defined as usual, apart from the possibility operator $\Diamond$, which we will not consider.

[2] Popper observed this problem for the first time. See Popper (1948b).

$$\neg_c\mathrm{I}\,\dfrac{Efq_i\,\dfrac{\neg_i\mathrm{E}\,\dfrac{\neg_i A \qquad [A]}{\bot_i}}{\bot_c}}{\neg_c A} \qquad\qquad \neg_i\mathrm{I}\,\dfrac{Efq_c\,\dfrac{\neg_c\mathrm{E}\,\dfrac{\neg_c A \qquad [A]}{\bot_c}}{\bot_i}}{\neg_i A}$$

According to Dummett, the impossibility of holding together these two logics suggests that classical and intuitionistic logics are based on different approaches toward logical validity and meaning, which are incoherent with each other. Indeed, Dummett distinguishes between:[3]

Conceptually trivial disagreements In these cases, using different labels, we can introduce all the logical constants of the two logics in the same language;

Conceptually deep disagreements In these cases the previous solution is not viable, and the choice between different logics reflects a decision between irreducibly different theories of meaning.

So, the impossibility of hosting together classical and intuitionistic notions just reflects the disagreement between a theory of meaning grounded on truth conditions and a theory of meaning grounded on assertibility conditions. Hence, every time the disagreement between two logics is deep, the meaning of all their constants is different. Moreover, the same holds even when there is an apparent agreement about a logical theorem between logics that are in deep disagreement. As an example, even though classicists and intuitionists seem to agree about the validity of $A \vdash A \vee B$, they do not mean the same thing with this expression. *A fortiori*, a change in meaning of both negation and disjunction is responsible for the disagreement about excluded middle.

2 *Prawitz's ecumenical system*

Prawitz (2015) explicitly refers to Dummett's position and argues that his conclusion holds only when classical logic is considered together with its standard theory of meaning based on truth conditions. In this case, the disagreement between classical and intuitionistic logic can only be deep, and it is impossible to merge the two systems. However, Prawitz objects that the same disagreement can be seen as trivial when an antirealist theory of meaning is provided for classical logic, and that this can be done with some tricks.

[3] See Dummett (1978, 285) and Dummett (1991, 193).

From the technical point of view, Prawitz proposes an ecumenical natural deduction system (let us call it **PEci**) for both classical and intuitionistic logic that overcomes the problem of the conflation of intuitionistic and classical negations. As remarked in Pimentel *et al.* (2021), Prawitz's system can be seen as an unusual way of looking at the so-called negative translation of **I** into **C**.[4]

- $p^\star = \neg_i \neg_i p$ for p atomic;
- $(A \wedge_c B)^\star = A^\star \wedge_i B^\star$;
- $(A \vee_c B)^\star = \neg_i(\neg_i A^\star \wedge_i \neg_i B^\star)$;
- $(\neg_c A)^\star = \neg_i A^\star$;
- $(A \supset_c B)^\star = \neg_i(A^\star \wedge_i \neg B^\star)$.

For this translation, it holds that:

Theorem 1 *For every set* $\Gamma \cup \{A\}$ *of* ***C*** *wff,* $\Gamma \vdash_C A$ *iff* $\Gamma^\star \vdash_I A^\star$

In Prawitz's ecumenical system, the classical and intuitionistic fragments share the same rules for $\wedge$, $\bot$ and $\neg$ (which are translated homophonically by the negative translation), while there are intuitionistic rules for $\supset_i$ and $\vee_i$, and classical rules for $\supset_c$, $\vee_c$ and for the classical atomic sentences P_c. The classical rules clearly derive from the corresponding clauses of the translation.

$$P_c\mathrm{I}\ \dfrac{\begin{matrix}[\neg P_i(t)]\\ \vdots\\ \bot\end{matrix}}{P_c(t)} \qquad P_c\mathrm{E}\ \dfrac{P_c(t) \qquad \neg P_i(t)}{\bot} \qquad \vee_c\mathrm{I}\ \dfrac{\begin{matrix}[\neg A, \neg B]\\ \vdots\\ \bot\end{matrix}}{A \vee_c B}$$

$$\vee_c\mathrm{E}\ \dfrac{A \vee_c B \qquad \neg A \qquad \neg B}{\bot} \qquad \supset_c\mathrm{I}\ \dfrac{\begin{matrix}[A, \neg B]\\ \vdots\\ \bot\end{matrix}}{A \supset_c B} \qquad \supset_c\mathrm{E}\ \dfrac{A \supset_c B \qquad A \qquad \neg B}{\bot}$$

In Prawitz's system, some results about intuitionistic and classical fragments are provable. First of all:[5]

Theorem 2 (Adequacy for **I**) *If no purely classical constants occur in* Γ *and* C*, then* $\Gamma \vdash_{PEci} C$ *iff* $\Gamma \vdash_I C$.

Theorem 3 (Adequacy for **C**) *If no purely intuitionistic constants occur in* Γ *and* C*, then* $\Gamma \vdash_{PEci} C$ *iff* $\Gamma \vdash_C C$.

[4] See Gödel (1986a, 287) (which saves only the theorems) and Gentzen (1969) (which saves both theorems and logical consequences):

[5] See Prawitz (2015).

Moreover, some results are provable about the entire language of **PEci**:[6]

(1) $\vdash_{PEci} (A \supset_i B) \supset_i (A \supset_c B)$

(2) $\vdash_{PEci} \neg\neg A \supset_c A$

(3) $\vdash_{PEci} \neg A \vee_c A$

(4) $\vdash_{PEci} (A \wedge (A \supset_i B)) \supset_i B$

(5) $\nvdash_{PEci} (A \wedge (A \supset_c B)) \supset_i B$

(6) $\Gamma \vdash_{PEci} C$ iff $\vdash_{PEci} \wedge\Gamma \supset_i C$

In addition to that, from the proof-theoretical point of view, Prawitz's system is normalizable,[7] and it can be converted into a sequent calculus in which Cut elimination holds.[8] Of course, there are other interesting properties of this system, but here these will be sufficient for our goals.

Prawitz concludes that it is possible for a classical logician to embrace an antirealist theory of meaning based on inferences, and that in this case the disagreement between them and the intuitionistic logician is to some extent trivial: the disagreements about the validity of excluded middle and Peirce's law are dissolved when they realize that the validity of these principles depends on whether classical or intuitionistic connectives are used, while the validity of double negation elimination holds only for classical sentences. As seen, this conclusion of Prawitz is explicitly in contradiction with Dummett's idea that the disagreement between classical and intuitionistic logic cannot be trivial. In particular, since in his ecumenical system the rules for negation are common to both classical and intuitionistic logic, while classical and intuitionistic disjunctions have their own respective rules, the disagreement about excluded middle is based on a difference in the meaning attached to 'or' and not to 'not'.

3 *The modal translation*

The negative translation of **C** into **I** is not the only translation possible that makes classical logic a subsystem of intuitionistic logic. There is another well-known class of translations, which map **I** into the modal logic **S4** (an extension of **C**).

[6] See Pimentel *et al.* (2021), section 3.1.
[7] See Pereira and Rodriguez (2017).
[8] See Pimentel *et al.* (2021), in which also other properties of this and related systems are proved.

Among them, we will consider Schütte's translation, which is a refinement of a similar translation proposed by Gödel:[9]

- $p^* = \Box p$ for p atomic;
- $(A \wedge_i B)^* = A^* \wedge_c B^*$;
- $(A \vee_i B)^* = A^* \vee_c B^*$;
- $(\neg_i A)^* = \Box \neg_c A^*$;
- $(A \supset_i B)^* = \Box(A^* \supset_c B^*)$.

For this translation, it holds that:

Theorem 4 *For every **I** wff C, $\vdash_I C$ iff $\vdash_{S4} C^*$*

Just like Prawitz's system is based on the negative translation of **C** into **I**, we can develop another ecumenical system inspired by the modal translation of **I** into **S4**. However, before displaying this system, let us investigate further the translation in itself.

First of all, let us observe that the translation * does not return all the theorems of **S4**, since it works with a proper subset of its well-formed-formulae. To make more precise this observation, let us consider the following definition:

Definition 5 (**I-S4** wff) *An **S4** wff is an **I-S4** wff iff*

- *each atomic formula p occurs as an immediate subformula of the formula $\Box p$;*
- *each conditional $A \supset B$ occurs as an immediate subformula of the formula $\Box(A \supset B)$;*
- *each negation $\neg A$ occurs as an immediate subformula of the formula $\Box\neg A$;*
- *$\Box$ is always applied to a negation, an implication or an atom.*

*We will use A_{IS4} to label formulae that are **I-S4** wff.*

For example, $\Box((p \wedge \Box q) \supset \Box\neg r)$ is an **I-S4** wff, while $\Box((p \wedge \Box q) \vee \Box\neg r)$ and $\Box((p \wedge \Box q) \supset \neg C)$ are not, because: in the first $\Box$ is applied to a disjunction, and in the second $\neg C$ is not an immediate subformula of $\Box\neg C$.

More compactly, we can define **I-S4** wff inductively as

$$A_{IS4} ::= \Box p | A_{IS4_1} \wedge A_{IS4_2} | A_{IS4_1} \vee A_{IS4_2} | \Box(A_{IS4_1} \supset A_{IS4_2}) | \Box\neg A_{IS4}$$

Remark *The translation * is onto **I-S4** wff.*

[9] See Gödel (1986a, 301) and Fitting (1969, 43).

Theorem 6 *The translation * takes all **I-S4** theorems. That is, for any A_{IS4}, if $\vdash_{S4} A_{IS4}$ then there is an **I** wff B such that $\vdash_I B$ and $B^* = A_{IS4}$.*

Proof. Given A_{IS4}, we obtain B just by removing each occurrence of $\Box$ in it. The translation * can only give back A_{IS4}, so that $B^* = A_{IS4}$. To show that $\vdash_I B$, consider that if $\nvdash_I B$, for theorem 4 we could obtain $\nvdash_{S4} B^*$ (that is $\nvdash_{S4} A_{IS4}$). $\Box$

So far we have considered only theorems, let us now focus on logical consequences and rules, and see whether they remain valid under modal translation. Susan Haack claims that Schütte's translation does not preserve derivability.[10] Unfortunately, she does not specify what she means with that claim, nor does she provide any proof or reference. An idea of what she is thinking about can be suggested by her treatment of the negative translation, for which she gives more details. She claims that the negative translation does not preserve derivability because the rule

$$\frac{\neg_i(A \wedge_i \neg_i B) \qquad A}{B}$$

is not derivable in **I**, even though it is the translation of classical Modus Ponens. Her observation is correct for Gödel's original translation, but not for Gentzen's translation, since it translates atomic classical sentences p as $\neg_i\neg_i p$. Accordingly, in Prawitz's ecumenical system Modus Ponens does not hold generally for classical implication, but holds when antecedent and consequent are classical, and in particular when they are classical atoms.

Schütte's modal translation clearly preserves deducibility in exactly the same way.[11] To show this, let us use the $\Diamond$-free fragment of the Ohinishi and Matsumoto's (1957) sequent calculus for propositional **S4**,[12] which consists in adding the rules

$$\frac{\Gamma, A \vdash \Delta}{\Gamma, \Box A \vdash \Delta}\,\mathrm{L}\Box \qquad \frac{\Box\Gamma \vdash A}{\Box\Gamma \vdash \Box A}\,\mathrm{R}\Box$$

to the propositional fragment of the classical sequent calculus **LK**. Let us prove that:[13]

[10] See Haack (1974, 96–97).

[11] This is not a new result, but since a proof of it is not easily found in the literature, we will briefly present one here.

[12] This fragment is complete and untouched by Routley's objection. See Routley (1975). See also Troelstra and Schwichtenberg (1996, sections 9.1 and 9.2).

[13] We use the notational ambiguity between $\vdash$ as the symbol for deducibility and as sequent symbol. This notational abuse cannot lead us astray, since the $\Diamond$-free Ohinishi and Matsumoto sequent calculus is adequate for this fragment of **S4**.

Theorem 7 *For every set $\Gamma \cup \{C\}$ of **I** wff, $\Gamma \vdash_I C$ iff $\Gamma^* \vdash_{S4} C^*$.*

Proof. $\Gamma \vdash_I C$ iff $\vdash_I \bigwedge_i \Gamma \supset_i C$, and so, by theorem 4, $\vdash_{S4} \Box((\bigwedge_i \Gamma)^* \supset_c C^*)$. $\vdash_{S4} \Box((\bigwedge_i \Gamma)^* \supset_c C^*)$ iff $\vdash_{S4} (\bigwedge_i \Gamma)^* \supset_c C^*$ (from left to right by L$\Box$ and Cut, from right to left by R$\Box$). Since Schütte's translation leaves untouched conjunctions (and disjunctions), we have $\vdash_{S4} \bigwedge_c \Gamma^* \supset_c C^*$, which is equivalent to $\Gamma^* \vdash_{S4} \bigwedge_c C^*$ (given the **S4**-rules for $\wedge$, $\supset$ and Cut). $\Box$

As a conclusion, we cannot interpret Haack's claim as stressing that Schütte's translation does not cover logical consequences or deducibility relations *stricto sensu*. If she is right, she has to mean something different. Interpreting Haack as claiming that the *-translation of **I**-rules are not derivable in **S4** makes her position more shareable. Indeed, if the translation of **NJ** rules for $\neg_i$ were derivable in **S4**, also the rules

$$\dfrac{\begin{array}{c}[\Box A]\\ \vdots \\ \bot\end{array}}{\Box\neg\Box A}\,\neg\mathrm{I}^* \qquad\qquad \dfrac{\Box\neg\Box A \qquad \Box A}{\bot}\,\neg\mathrm{E}^*$$

should be derivable *a fortiori*, being special cases of that translation.

However, these rules make possible to derive $\Box\neg\Box A$ from $\neg\Box A$ (which is clearly unacceptable in **S4**), via the derivation[14]

$$\dfrac{\dfrac{\neg\Box A \qquad [\Box A]^1}{\bot}\,\neg\mathrm{E}}{\Box\neg\Box A}\,\neg\mathrm{I}^*_1$$

As a consequence, Haack's observation is right if interpreted in this sympathetic way.

However, it should be observed that the problematic application of these rules takes as premise an **S4** wff that is not an **I**-**S4** wff (that is, $\neg\Box A$). This is not a contingent fact due to the example we selected. Indeed, it can be proved that if we restrict the translated rules so that they operate only on **I**-**S4** wff, they become derivable in **S4**. Nonetheless, in order to show that, we need to focus on sequent calculus, since this proof system allows restriction on the wff of the context as well.

[14] Just as a clarification, the first rule applied in the derivation is the rule $\neg$E for **S4**, it is not the one obtained by translating $\neg$E for **I**.

Theorem 8 *The translation * of the rules of the propositional fragment of **LJ** is derivable in the $\Diamond$-free propositional fragment of Ohinishi and Matsumoto's sequent calculus for **S4**. That is, for every application of a propositional rule in **LJ***

$$\dfrac{\Gamma \vdash C \qquad \Delta \vdash D}{\Theta \vdash E}\, Rule$$

its *-translation

$$\dfrac{\Gamma^* \vdash C^* \qquad \Delta^* \vdash D^*}{\Theta^* \vdash E^*}\, Rule^*$$

*is derivable in the $\Diamond$-free propositional fragment of Ohinishi and Matsumoto's sequent calculus for **S4**.*

Before seeing the proof of this theorem, a specification is needed. We stated it explicitly about applications of rules, to make clear that what we obtain is not the derivability of the translated rules in their complete generality. In other words, the translations of **I**-rules should not be generalized as to apply to **S4** wff that are not **I**-**S4** wff. Without this specification, the following translation in sequent calculus of the unacceptable natural deduction derivation of $\Box\neg\Box A$ from $\neg\Box A$ would be a counterexample to the soundness of our translation for rules:

$$\dfrac{\dfrac{\dfrac{\dfrac{A \vdash A}{\Box A \vdash A}\,\mathrm{L}\Box}{\Box A \vdash \Box A}\,\mathrm{R}\Box}{\Box A, \neg\Box A \vdash}\,\mathrm{L}\neg}{\neg\Box A \vdash \Box\neg\Box A}\,\mathrm{R}\neg^*$$

On the contrary, this derivation cannot be accepted, because the application of $\mathrm{R}\neg^*$ is not the translation of any application of intuitionistic $\mathrm{R}\neg$. Another way of seeing this specification is that the translation of the context should be considered as well.[15] From the practical point of view, this means restricting the application of the translated rules to **I-S4** wff only. In other words, the modal translation preserves derivability just like the negative translation does, but in both cases we have to be careful in translating also the sentences belonging to the context. We will see that in some (but not in all!) cases this restriction about the context can be dropped.

In order to prove theorem 8, we will also need the following lemma:

Lemma 9 *For every **I**-**S4** wff C_{IS4}, $C_{IS4} \vdash_{S4} \Box C_{IS4}$*

[15] As made clear by the application of * to them as well.

Proof. The proof is by induction on the complexity of C_{IS4}.[16] Moreover, consider that the lemma holds only for **I-S4** wff and this restricts both the base case and some of the induction steps.

Base case ($C_{IS4} = \Box A$) For Axiom 4, $\Box A \vdash_{S4} \Box\Box A$;

Case $C_{IS4} = A_{IS4} \wedge B_{IS4}$

$$\dfrac{\mathrm{Cut}\ \dfrac{\dfrac{\dfrac{\text{Ind. Hypo.}}{A_{IS4} \vdash \Box A_{IS4}} \quad \dfrac{\text{Ind. Hypo.}}{B_{IS4} \vdash \Box B_{IS4}}}{A_{IS4}, B_{IS4} \vdash \Box A_{IS4} \wedge \Box B_{IS4}}\,\mathrm{R}\wedge \qquad \begin{array}{c}\vdots\\ \Box A_{IS4} \wedge \Box B_{IS4} \vdash \Box(A_{IS4} \wedge B_{IS4})\end{array}}{A_{IS4}, B_{IS4} \vdash \Box(A_{IS4} \wedge B_{IS4})}}{A_{IS4} \wedge B_{IS4} \vdash \Box(A_{IS4} \wedge B_{IS4})}\,\mathrm{L}\wedge$$

Case $C_{IS4} = A_{IS4} \vee B_{IS4}$

$$\dfrac{\mathrm{Cut}\ \dfrac{\dfrac{\text{Ind. Hypo.}}{A_{IS4} \vdash \Box A_{IS4}} \quad \begin{array}{c}\vdots\\ \Box A_{IS4} \vdash \Box(A_{IS4} \vee B_{IS4})\end{array}}{A_{IS4} \vdash \Box(A_{IS4} \vee B_{IS4})} \quad \mathrm{Cut}\ \dfrac{\dfrac{\text{Ind. Hypo.}}{B_{IS4} \vdash \Box B_{IS4}} \quad \begin{array}{c}\vdots\\ \Box B_{IS4} \vdash \Box(A_{IS4} \wedge B_{IS4})\end{array}}{B_{IS4} \vdash \Box(A_{IS4} \vee B_{IS4})}}{A_{IS4} \vee B_{IS4} \vdash \Box(A_{IS4} \vee B_{IS4})}\,\mathrm{L}\vee$$

Case $C_{IS4} = \Box(A_{IS4} \supset B_{IS4})$ For Axiom 4, $\Box(A_{IS4} \supset B_{IS4}) \vdash_{S4} \Box\Box(A_{IS4} \supset B_{IS4})$;[17]

Case $C_{IS4} = \Box\neg A_{IS4}$ For Axiom 4, $\Box\neg A_{IS4} \vdash_{S4} \Box\Box\neg A_{IS4}$. $\Box$

In the end, let us now prove theorem 8.

Proof. We will show that the rules of **LJ** are derivable in **S4** when they are formulated using only **I-S4** wff. From this, it clearly follows that for every application of an **LJ**-rule, its *-translation is derivable in **S4**.

For the structural rules, the result is obvious, since the translation of the **I**-rules are just special cases of their **S4** analogues. So, let us consider the operational rules.

R$\wedge_{IS4}$ and L$\wedge_{IS4}$

$$\dfrac{\Gamma \vdash A_{IS4} \qquad \Delta \vdash B_{IS4}}{\Gamma, \Delta \vdash A_{IS4} \wedge B_{IS4}}\,\mathrm{R}\wedge_{IS4} \qquad \dfrac{\Gamma, A_{IS4} \vdash C}{\Gamma, A_{IS4} \wedge B_{IS4} \vdash C}\,\mathrm{L}\wedge_{IS4}$$

[16] In general the induction hypotheses are justified because A_{IS4} and B_{IS4} are **I-S4** wff.
[17] Of course, $A_{IS4} \supset B_{IS4}$ is not an **I-S4** wff, so we do not have to consider it.

The rules are special cases of the **S4**-rules for conjunction, and so are derivable even without any restriction on the contexts (Γ and Δ). Consider that $A_{IS4} \wedge B_{IS4}$ is an **I-S4** wff.

R$\vee_{IS4}$ **and L**$\vee_{IS4}$ Idem.

R$\supset_{IS4}$

$$\frac{\Gamma_{IS4}, A_{IS4} \vdash B_{IS4}}{\Gamma_{IS4} \vdash \Box(A_{IS4} \supset B_{IS4})}\,\mathrm{R}{\supset_{IS4}}$$

Here the restriction on the context is needed, since we need an application of lemma 9.

$$\dfrac{\dfrac{\text{lemma 9}}{\gamma_{IS4} \vdash \Box\gamma_{IS4}} \qquad \dfrac{\dfrac{\dfrac{\Gamma_{IS4}, A_{IS4} \vdash B_{IS4}}{\Box\Gamma_{IS4}, A_{IS4} \vdash B_{IS4}}\,\mathrm{L}\Box}{\Box\Gamma_{IS4} \vdash A_{IS4} \supset B_{IS4}}\,\mathrm{R}{\supset}}{\Box\Gamma_{IS4} \vdash \Box(A_{IS4} \supset B_{IS4})}\,\mathrm{R}\Box}{\Gamma_{IS4} \vdash \Box(A_{IS4} \supset B_{IS4})}\,\text{Cuts}$$

L$\supset_{IS4}$

$$\frac{\Gamma \vdash A_{IS4} \qquad \Delta, B_{IS4} \vdash C}{\Gamma, \Delta, \Box(A_{IS4} \supset B_{IS4}) \vdash C}\,\mathrm{L}{\supset_{IS4}}$$

Note that $\Box(A_{IS4} \supset B_{IS4})$ is an **I-S4** wff. This rule can be derived in **S4** by

$$\dfrac{\dfrac{\Gamma \vdash A_{IS4} \qquad \Delta, B_{IS4} \vdash C}{\Gamma, \Delta, A_{IS4} \supset B_{IS4} \vdash C}\,\mathrm{L}{\supset}}{\Gamma, \Delta, \Box(A_{IS4} \supset B_{IS4}) \vdash C}\,\mathrm{L}\Box$$

R$\neg_{IS4}$

$$\frac{\Gamma_{IS4}, A_{IS4} \vdash}{\Gamma_{IS4} \vdash \Box\neg A_{IS4}}\,\mathrm{R}{\neg_{IS4}}$$

As for $\mathrm{R}{\supset}_{IS4}$, the restriction on the context is needed, since we need an application of lemma 9.

$$\dfrac{\dfrac{\text{lemma 9}}{\gamma_{IS4} \vdash \Box\gamma_{IS4}} \qquad \dfrac{\dfrac{\dfrac{\Gamma_{IS4}, A_{IS4} \vdash}{\Gamma_{IS4} \vdash \neg A_{IS4}}\,\mathrm{R}\neg}{\Box\Gamma_{IS4} \vdash \neg A_{IS4}}\,\mathrm{L}\Box}{\Box\Gamma_{IS4} \vdash \Box\neg A_{IS4}}\,\mathrm{R}\Box}{\Gamma_{IS4} \vdash \Box\neg A_{IS4}}\,\text{Cuts}$$

L$\neg_{IS4}$

$$\dfrac{\Gamma \vdash A_{IS4}}{\Gamma, \Box\neg A_{IS4} \vdash}\,\mathrm{R}\neg_{IS4} \quad \text{is derived by} \quad \mathrm{R}\neg\,\dfrac{\dfrac{\Gamma \vdash A_{IS4}}{\Gamma, \neg A_{IS4} \vdash}}{\Gamma, \Box\neg A_{IS4} \vdash}\,\mathrm{L}\Box$$

$\Box$

The previous results suggest that the modal translation is a good base to develop another ecumenical system, alternative to Prawitz's one.

4 *Ecumenical system based on S4*

While Prawitz's system starts as an intuitionistic system, which then is extended with rules for classical connectives, we will start with **LK**, and extend this system with rules for intuitionistic constants. Since Schütte's modal translation associates intuitionistic atoms with modalised atoms, it is natural to adapt L$\Box$ and R$\Box$ as rules for the intuitionistic atoms. The principal formula in both LP_i and RP_i is of course P_i, which substitutes P_c, occurring in place of the unmodalised active formula of the premise of L$\Box$ and R$\Box$. There is some ambiguity on how to interpret the restrictions on the context in R$\Box$: surely, the restriction that there should be only one sentence on the succedent remains the same for RP_i, but the requirement that all sentences in the antecedent should be of the form $\Box C$ could be interpreted in different ways. Here we will require that they are intuitionistic atoms P_i or in them the outermost connective is $\supset_i$ or $\neg_i$. We will call them purely intuitionistic constants. Moreover, not surprisingly we will see that purely classical constants are only $\supset_c$, $\neg_c$ and P_c, while the other constants are shared between the classical and intuitionistic fragments. Indicating with Γ^i this restriction, we can formulate the rules for intuitionistic atoms as follows:

$$\frac{\Gamma, P_c \vdash \Delta}{\Gamma, P_i \vdash \Delta}\,\mathrm{L}P_i \qquad \frac{\Gamma^i \vdash P_c}{\Gamma^i \vdash P_i}\,\mathrm{R}P_i$$

Since Schütte's modal translation translates homophonically conjunction and disjunction, the rules of **LK** for these connectives can be assumed common for both the intuitionistic and classical fragment. On the contrary, negation and implication require new rules, being translated non-homophonically. Let us consider negation first. Since $\neg_i A$ is translated with $\Box\neg_c A$, we act as to introduce (to the left and to the right) a modally loaded negation. That is, given the restriction on the rules of **S4** for $\Box$, we must have only the negated formula in the succedent of $\mathrm{R}\neg_i$ and the antecedent must have the same restriction of $\mathrm{R}P_i$: only intuitionistic constants can occur in it as outermost connectives. In other words, we have to mimic the derivation

$$\frac{\dfrac{\Box\Gamma, A \vdash}{\Box\Gamma \vdash \neg A}\,\mathrm{R}\neg}{\Box\Gamma \vdash \Box\neg A}\,\mathrm{R}\Box$$

As for $\mathrm{L}\neg_i$, since there are no restrictions in **S4** for the introduction of $\Box$ on the left, we can stay with the usual rules. In summary, the rules for $\neg_i$ are:

$$\frac{\Gamma \vdash A, \Delta}{\Gamma, \neg_i A \vdash \Delta}\,\mathrm{L}\neg_i \qquad \frac{\Gamma^i, A \vdash}{\Gamma^i \vdash \neg_i A}\,\mathrm{R}\neg_i$$

The rules for implication can be treated in essentially the same way, so to obtain:

$$\frac{\Gamma \vdash A, \Delta \qquad \Theta, B \vdash \Lambda}{\Gamma, \Theta, A \supset_i B \vdash \Delta, \Lambda}\,\mathrm{L}{\supset_i} \qquad \frac{\Gamma^i, A \vdash B}{\Gamma^i \vdash A \supset_i B}\,\mathrm{R}{\supset_i}$$

Even though these rules are inspired to the sequent system for **S4**, when they are added to the rules of **LK** for conjunction and disjunction (together with the structural ones), the system obtained clearly resembles Maehara's intuitionistic system **LJ'**.[18] This fact will be useful in proving its adequacy for intuitionistic logic. In order to obtain our ecumenical system, we just need to add these rules to the full system **LK** in which the rules for implication and negation are reformulated with $\supset_c$ and $\neg_c$, to stress their classical nature. Let us label **MEci** this new ecumenical system based on the modal translation.[19]

[18] See Takeuti (1987, 52).

[19] Matteo Tesi has developed independently a (labeled) sequent calculus for an ecumenical system based on the modal translation of **I** into **S4**. While there are some formal differences between his system and mine, they are essentially equivalent. See Tesi (2023). I thank him for kind and interesting discussions on these issues.

As a first observation, we can see that the usual derivation of $\neg_i A$ from $\neg_c A$

$$\dfrac{\dfrac{A \vdash A}{\neg_c A, A \vdash}\,\mathrm{L}\neg_c}{\neg_c A \vdash \neg_i A}\,\mathrm{R}\neg_i$$

is blocked by the restriction imposed by R$\neg_i$.

Let us now prove Cut elimination for this ecumenical system, which will be useful to prove various of its other properties.

Theorem 10 (Cut elimination) *If $\Gamma \vdash \Delta$ is provable in our ecumenical system **MEci**, there is a proof of this sequent that does not use Cut.*

Proof. The proof is as usual: by induction on its degree and on its rank, we show that the top-most Cut of any derivation can be removed. The majority of the cases are already provided by the Cut elimination for **LK**. Let us consider the remaining cases, that is the ones regarding P_i, $\supset_i$ and $\neg_i$.

In the principal cases, both Cut-formulae are active in the premises. In this case, the degree of the Cut-formula can be reduced. As an example

$$\dfrac{\dfrac{\Gamma^i, A \vdash B}{\Gamma^i \vdash A \supset_i B}\,\mathrm{R}{\supset_i} \qquad \dfrac{\Delta \vdash A, \Theta \qquad \Lambda, B \vdash \Xi}{\Delta, \Lambda, A \supset_i B \vdash \Theta, \Xi}\,\mathrm{L}{\supset_i}}{\Delta, \Lambda, \Gamma^i \vdash \Theta, \Xi}\,\mathrm{Cut}$$

can be reduced to

$$\dfrac{\dfrac{\Delta \vdash A, \Theta \qquad \Gamma^i, A \vdash B}{\Delta, \Gamma^i \vdash \Theta, B}\,\mathrm{Cut} \qquad \Lambda, B \vdash \Xi}{\Delta, \Lambda, \Gamma^i \vdash \Theta, \Xi}\,\mathrm{Cut}$$

and the same holds for P_i and $\neg_i$.

The non-principal cases are a little trickier. Of course at most one of the Cut-formulae can be active in the premise of the Cut rule. If the Cut-formula is active in the right premise then the derivation must be something like

$$\dfrac{\Lambda \vdash \Theta, \neg_i A \qquad \dfrac{\Gamma \vdash A, \Delta}{\Gamma, \neg_i A \vdash \Delta}\,\mathrm{L}\neg_i}{\Lambda, \Gamma \vdash \Theta, \Delta}\,\mathrm{Cut}$$

Note that $\neg_i A$ cannot be active in the left premise, since otherwise this would be a principal case (already considered). So, $\neg_i A$ is already present in one of the premises of the rule that has $\Lambda \vdash \Theta, \neg_i A$ as its conclusion. Moreover, this rule cannot be L$\supset_i$, L$\neg_i$ or LP_i, since these have only their respective principal formula in the succedent of their conclusion. It follows that there is no restriction on the context that could prevent a permutation upward of the Cut on the left. This last observation holds in general for all non-principal cases in which the Cut-formula is active in the right premise (when L$\supset_i$, L$\neg_i$, LP_i, or also any classical rule is applied as last rule on the right). In conclusion, since $\neg_i A$ already occurs in one of the premises of the rule that has the left premise of the Cut as its conclusion, and this rule cannot have any restriction on the context, we can permute the derivation so to reduce the left rank of the Cut. Other cases are essentially identical.

Let us now consider the cases in which the Cut-formula is not active on the right premise of Cut. Let us take for example the following case in which R$\supset_i$ is the last rule applied in the derivation of the right premise of Cut, but (obviously) it does not introduce the Cut-formula:

$$\dfrac{\Delta \vdash \Theta, C^i \qquad \dfrac{C^i, \Gamma^i, A \vdash B}{C^i, \Gamma^i \vdash A \supset_i B}\,\mathrm{R}\!\supset_i}{\Delta, \Gamma^i \vdash \Theta, A \supset_i B}\,\mathrm{Cut}$$

Given the restrictions on R$\supset_i$, the Cut-formula must have an intuitionistic constant as its outermost symbol. However, this is not enough to legitimate the permutation of Cut upward. To justify this permutation, we have to reason by cases on the last rule applied on the derivation of the left premise of Cut.

If the Cut-formula is not active in the left premise either, then the last rule applied on the left cannot be R$\supset_i$, R$\neg_i$ or RP_i. Indeed these rules have only their principal formula in the succedent of their conclusion. Hence, the reduction is identical to the one already seen for Cut-formulae active on the right but not on the left, and eventual restrictions on the last rule applied on the right do not cause any trouble, the reduction being on the left. As an example, if the last rule applied in the derivation of the left premise of the Cut is R$\supset_c$, the derivation is

$$\dfrac{\dfrac{\Delta D \vdash E, \Theta, C^i}{\Delta \vdash D \supset_c E, \Theta, C^i}\,\mathrm{R}\!\supset_c \qquad \dfrac{C^i, \Gamma^i, A \vdash B}{C^i, \Gamma^i \vdash A \supset_i B}\,\mathrm{R}\!\supset_i}{\Delta, \Gamma^i \vdash \Theta, A \supset_i B, D \supset_c E}\,\mathrm{Cut}$$

and it can be reduced to

$$
\dfrac{\Delta, D \vdash E, \Theta, C^i \qquad \dfrac{C^i, \Gamma^i, A \vdash B}{C^i, \Gamma^i \vdash A \supset_i B}\,\mathrm{R}{\supset_i}}{\dfrac{\Delta, \Gamma^i, D \vdash E, \Theta, A \supset_i B}{\Delta, \Gamma^i \vdash \Theta, A \supset_i B, D \supset_c E}\,\mathrm{R}{\supset_c}}\,\mathrm{Cut}
$$

This permutation reduces the left rank of the Cut.

If the Cut-formula is active on the left, let us consider the case in which $C^i = \neg_i D$:

$$
\dfrac{\dfrac{\Delta^i, D \vdash}{\Delta^i \vdash \neg_i D}\,\mathrm{R}\neg_i \qquad \dfrac{\neg_i D, \Gamma^i, A \vdash B}{\neg_i D, \Gamma^i \vdash A \supset_i B}\,\mathrm{R}{\supset_i}}{\Delta^i, \Gamma^i \vdash A \supset_i B}\,\mathrm{Cut}
$$

This derivation can be reduced to

$$
\dfrac{\dfrac{\dfrac{\Delta^i, D \vdash}{\Delta^i \vdash \neg_i D}\,\mathrm{R}\neg_i \qquad \neg_i D, \Gamma^i, A \vdash B}{\Delta^i, \Gamma^i, A \vdash B}\,\mathrm{Cut}}{\Delta^i, \Gamma^i \vdash A \supset_i B}\,\mathrm{R}{\supset_i}
$$

In this way, the right rank of Cut is reduced. Note that the fact that Δ^i suits the restrictions for R$\supset_i$ is needed for the reduction to be 'legal'.

This reduction procedure can be generalized to all cases in which the Cut-formula is active on the left but not on the right. Indeed, if the last rule applied on the right has no restriction on the context, the reduction is obviously unproblematic (the sub-derivation of the left premise is untouched by the reduction). If the last rule applied on the right has restrictions, it must be R$\supset_i$, R$\neg_i$ or RP_i. In all these cases, the Cut-formula must have an intuitionistic connective as its outermost logical term (given the restriction on Γ^i). So, if the Cut-formula is active on the left, the last rule applied on the left can only be R$\supset_i$, R$\neg_i$ or RP_i. Hence, the last rules applied on the left and on the right must have the same restrictions on the context (that is, the fact that Δ^i is well-suited for the restriction of R$\supset_i$ in our example is not causal) and the derivation can be reduced.[20]

[20] Let us notice that the restriction on the context of R$\supset_i$, R$\neg_i$ and RP_i is what makes possible this reduction. As an example, if we were to liberalize this restriction allowing not only purely intuitionistic constants to occur as outermost constants in Γ^i, but also non-purely classical terms (like $\wedge$ and $\vee$), we would need a reduction for

The base cases with axioms as left or right premises of the Cut are obvious. Hence, the theorem holds.

□

Let us now prove adequacy of the classical and intuitionistic fragments of our ecumenical system for their respective logics.

Theorem 11 (Adequacy for **I**) *If no purely classical constants occur in* Γ *and* C*, then* $\Gamma \vdash_{MEci} C$ *iff* $\Gamma \vdash_I C$.

Proof. To show the completeness of **MEci**, let us consider fist of all that if $\Gamma \vdash_I C$, then $\Gamma^* \vdash_{S4} C^*$. Now, if $\Gamma^* \vdash_{S4} C^*$, then this sequent is provable in Ohinishi and Matsumoto's system. Since Cut is eliminable in their system, let us take a Cut-free proof of this sequent. In this derivation, since Γ^* and C^* are **I-S4** wff we can be sure that

- If $\Box C$ occurs in it, then C is an atom, an implication or a negation;
- If a negation, an implication, or an atom occurs unmodalised in it, it is modalised somewhere below by an application of a $\Box$-rule.

If this were not the case, in the conclusion we would have formulae that are not **I-S4** wff (remember that the derivation is Cut-free).

In order to convert the derivation of $\Gamma^* \vdash C^*$ into a **MEci**-derivation of $\Gamma \vdash C$, we just need to:

- Move all applications of $\Box$-rules upward, so that as soon as an atom, an implication, or a negation is introduced, it immediately gets modalised in the next step of the derivation;[21]
- Fuse together $\neg$ and $\supset$-rules with $\Box$-rules, in the following way[22]

$$\dfrac{\dfrac{\Delta \vdash \Theta, C^i}{\Delta \vdash \Theta, C^i \vee D^i}\,R\vee \qquad \dfrac{C^i \vee D^i, \Gamma^i, A \vdash B}{C^i \vee D^i, \Gamma^i \vdash A \supset_i B}\,R\supset_i}{\Delta, \Gamma^i \vdash \Theta, A \supset_i B}\,\text{Cut}$$

However, it is far from obvious how to reduce this derivation, since R$\vee$ has no restriction on the context. Hence, it is key that when a rule imposes a restriction on the occurrence as outermost logical constant in the formulae of its context, it requires the occurrence of those connectives for which the R-rules have restrictions themselves.

[21] Observe that the restrictions on the context for the applications of R$\Box$ do not pose any issue for this permutation.

[22] The other cases are identical.

$$\dfrac{\dfrac{\Box\Gamma, A \vdash}{\Box\Gamma \vdash \neg_i A}\,\mathrm{R}\neg_i}{\Box\Gamma \vdash \Box\neg_i A}\,\mathrm{R}\Box \quad \rightsquigarrow \quad \dfrac{\Box\Gamma, A \vdash}{\Box\Gamma \vdash \Box\neg_i A}\,\mathrm{R}\Box\neg_i$$

- Apply the following reverse translation

  - $p^\dagger = P_c$ for p atomic;
  - $(\Box p)^\dagger = P_i$ for p atomic;
  - $(A \wedge B)^\dagger = A^\dagger \wedge B^\dagger$;
  - $(A \vee B)^\dagger = A^\dagger \vee B^\dagger$;
  - $(\Box\neg A)^\dagger = \neg_i A^\dagger$;
  - $(\Box(A \supset B))^\dagger = A^\dagger \supset_i B^\dagger$.

  Note that this translation is complete for **I**-**S4** wff, which are the only formulae left in the derivation after the application of the previous step, apart from atomic sentences p that occur unmodalised and are translated as classical atoms. More precisely, the translation is recursive and has two basic conditions: if an atom occurs modalised, it is translated as an intuitionistic atom; otherwise, it is translated as a classical atom.

This procedure delivers a valid proof in **MEci**. In particular, observe that the conditions on the applicability of $\Box$-rules and the fact that the modal translation is onto **I**-**S4** wff warrant that the conditions for the applicability of RP_i, R$\supset_i$ and R$\neg_i$ hold.

Let us now prove soundness of **MEci** for **I**, when purely classical connectives do not occur in the endsequent. Given Cut elimination for **MEci**, we can assume to have a Cut-free derivation for $\Gamma \vdash_{MEci} C$. Of course, $\supset_c$ and $\neg_c$ cannot occur in the Cut-free proof, since otherwise they would occur in the conclusion as well. Moreover, if we identify P_i and P_c removing all subscripts, and we erase all applications of P_i-rules, we obtain a sound derivation of $\Gamma \vdash C$ in Maehara's system **LJ'**. $\Box$

Theorem 12 (Adequacy for **C**) *If no purely intuitionistic constants occur in* Γ *and* C*, then* $\Gamma \vdash_{MEci} C$ *iff* $\Gamma \vdash_C C$.

Proof. Completeness of **MEci** for **C** is obvious, since all the rules of **LK** are also rules of **MEci**. Soundness is just a little less obvious. For Cut elimination, let us take a Cut-free derivation of $\Gamma \vdash C$ in **MEci**. In this derivation, no rule for purely intuitionistic constants can be applied, and so this derivation uses only **LK**-rules. $\Box$

While in Prawitz's ecumenical system **PEci** intuitionistic and classical logics share their rules for $\bot$, $\neg$ and $\wedge$, in our system **MEci**, based on the modal translation, these logics share rules for $\wedge$ and $\vee$. As for $\bot$, **MEci** does not need to adopt explicit rules for this constant, since it is developed within a sequent setting. Anyway, the fact that structural rules are common to both the classical and intuitionistic fragments clearly highlights that the inferential meaning of absurdity is common to both logics. Hence, **PEci** consider the disagreement between classicists and intuitionists as grounded on a difference in the meaning of implication and disjunction, while **MEci** consider the disagreement between classicists and intuitionists as grounded on a difference in the meaning of implication and negation. Both systems share the position that the disagreement between these two logics is trivial, that is solvable by simply specifying the meaning of the constants in use.

Let us briefly compare the positions of these two ecumenical systems about the meaning of the logical terms. Regarding the excluded middle, **MEci** suggests that the disagreement between classicists and intuitionists is grounded on a difference in the meaning of negation, as opposed to **PEci**, which blames disjunction for the disagreement. Moreover, we can compare the two systems by considering the properties already proved for the whole language of Prawitz's system.[23] For **MEci**, the following properties are provable:

(1) $\vdash_{MEci} (A \supset_i B) \supset_i (A \supset_c B)$

(2) $\vdash_{MEci} \neg_c \neg_c A \supset_c A$

(3) $\vdash_{MEci} \neg_c A \vee A$

(4) $\vdash_{MEci} (A \wedge (A \supset_i B)) \supset_i B$

(5) $\vdash_{MEci} (A \wedge (A \supset_c B)) \supset_i B$

(6) $\Gamma \vdash_{MEci} C$ iff $\vdash_{MEci} \wedge\Gamma \supset_i C$

Points 1, 2 and 3 mark an essential agreement between the two ecumenical systems. Of course, there are some differences in which constants are considered responsible for the validity of the purely classical results, but this does not result in a disagreement about their validity. This is not surprising, since both systems are designed to have classical and intuitionistic subsystems.

Point 5 is interesting because marks an explicit disagreement between **MEci** and **PEci**, it being provable in the first but not in the second. Moreover, since what is really at issue with this point is the validity of Modus Ponens for classical

[23] See page 35.

implication, we can observe that $A, A \supset_c B \vdash_{MEci} B$ even though in general $A, A \supset_c B \nvdash_{PEci} B$. So MP does not hold for classical implication in **PEci**, but it does in **MEci**. On the contrary, since in **MEci** there are no restrictions on L$\supset_i$, MP holds for intuitionistic conditional in both **PEci** and **MEci**.

Points 4 and 6 can be discussed together, since they deal essentially with deduction theorem. Let us start with point 6. It apparently remarks an agreement between **PEci** and **MEci**, since in both of them a sentence C is provable from a set of sentences Γ if and only if the intuitionistic implication $\Gamma \supset_i C$ is provable. However, while in **PEci** the relation of derivation is intuitionistic in general (becoming classical only when the vocabulary is restricted to classical constants), in **MEci** the relation of derivation is clearly classical. Indeed:

- $\Gamma, C \vdash_{MEci} D$ iff $\Gamma \vdash_{MEci} C \supset_c D$
- For some Γ, C and D, it holds that $\Gamma, C \vdash_{MEci} D$, even though $\Gamma \nvdash_{MEci} C \supset_i D$.

The first point is obvious. As for the second one, consider $\vdash_{MEci} (A \wedge (A \supset_c B)) \supset_i B$ (which for point 5 is provable), and MP for classical implication $A, A \supset_c B \vdash_{MEci} B$ (which, as seen, holds in **MEci**). Consider then $A \supset_c B \vdash A \supset_i B$ or $A \vdash (A \supset_c B) \supset_i B$, which clearly cannot be derivable in **MEci**, because of the restrictions on the applicability of R$\supset_i$. So, contrary to what point 6 may suggest, logical deduction in **MEci** is classical and not intuitionistic, as opposed to the intuitionistic-based ecumenic system **PEci**. In conclusion, point 4 is a point of agreement between **PEci** and **MEci**, but it is equivalent to the validity of MP for intuitionistic implication only in **PEci** – its equivalent in **MEci** being $(A \wedge (A \supset_i B)) \supset_c B$, which obviously holds.

5 *Conclusion*

We have seen that, remaining faithful to the semantical tradition, we can explain the disagreement between classicists and intuitionists as grounded on a difference in the meaning of disjunction, negation, or both of them. A good criterion to discern between these options has still to be proposed. Of course, we can distinguish between a deep disagreement in which classical and intuitionistic logicians endorse radically different and incompatible theories of meaning, like the ones considered by Dummett, and trivial disagreements in which those logicians endorse the same general theory of meaning, but chose different sets of connectives. While the disagreement between classicists and intuitionists in the last case is trivial, as suggested by the possibility of developing ecumenical systems, it is far from clear how to treat the disagreement between different ecumenical systems themselves. In other words, if the disagreement between classicists and intuitionists about the validity of $\neg C \vee C$ can be seen as trivial and just regarding which disjunction or negation to use, what can we say about the disagreement between proponents of **PEci** who endorse $(A \wedge (A \supset_c B)) \supset_i B$ and proponents of **MEci** who reject it? Apparently, the possibility of developing an ecumenical system is just the starting point for dealing with (so-called) *trivial* logical disagreements.

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