New problems for Tennant's definition of harmony

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Abstract

Steinberger proposes as a counterexample to Tennant's *harmony criterion* a degenerate quantifier that makes it possible to prove consequences such as $A(a) \vDash A(b)$ for every propositional function and pair of terms. Tennant suggests an answer that, according to Steinberger, is inadequate for the purpose. In this paper, it is shown that Steinberger's counterexample works only within a finitary framework, and that, moreover, it rests on a controversial interpretation of Tennant's *criterion*. Another deepened interpretation of the same *criterion* is suggested and defended. It is then proved that this new reading solves Steinberger's counterexample but leads to an unacceptable weakening of the logical system.

Keywords: Harmony; Tennant; Steinberger; Counterexample; Existential; Rule-Strength

1 Tennant's *criterion*

Tennant proposes a *criterion* to distinguish which pairs of rules of introduction and rules of elimination (I and E-rules) really characterize logical constants, and to exclude degenerate connectives like Prior's *tonk*.[1] The aims of this paper are:

- To show that a *tonk*-like quantificational counterexample proposed by Steinberger against this *criterion* relies on an approximate interpretation of some side clauses already present in Tennant's work, and that this counterexample is blocked by a more refined interpretation of such clauses;
- To show that this refined interpretation of the side clauses causes problems just slightly less severe to Tennant's *criterion*, making unacceptable some rules usually believed to be uncontroversial.

So, a general defence of Tennant's *criterion* is not only beyond the scope of this paper, but even questioned by one of its conclusions.

1.1 Definition of $\underline{\mathbf{H}}$*armony*

Following Tennant, we will distinguish between '$\underline{\mathrm{H}}$*armony*', which is the complete *criterion* of acceptability, and '$\underline{\mathrm{h}}$*armony*', which is just the first part of it.[2] For a pair of sets of rules for $\circledS$ to be in $\underline{\mathrm{H}}$*armony*, two properties must hold:[3]

$\underline{\mathbf{h}}$*armony* The pair must be in $\underline{\mathrm{h}}$*armony*, that is:

1. $A\circledS B$ is the strongest conclusion possible under the conditions described by $\circledS$I. Moreover, in order to show this:
   (a) one needs to exploit all the conditions described by $\circledS$I;
   (b) one needs to make full use of $\circledS$E;
   (c) one may not make any use of $\circledS$I.
2. $A\circledS B$ is the weakest major premise possible under the conditions described by $\circledS$E. Moreover, in order to show this:
   (a) one needs to exploit all the conditions described by $\circledS$E;
   (b) one needs to make full use of $\circledS$I;
   (c) one may not make any use of $\circledS$E.

[1] Prior [1960].

[2] The most complete formulations of Tennant's notion of *harmony* can be found in Tennant [1997] and especially in Tennant [201X].

[3] We treat only binary constants.

maximality Given ⓈI, ⓈE must be the strongest E-rule in $\underline{\text{h}}$*armony* with ⓈI and, given ⓈE, ⓈI must be the strongest I-rule in $\underline{\text{h}}$*armony* with ⓈE.

To a first approximation, the notions of strength of a proposition and strength of a rule are enough clear. A proposition is the strongest having such-and-such property if it entails every proposition having the same property; it is the weakest if it is entailed by every proposition having the same property. A rule is at least as strong as another if it enables us to derive, from the same assumptions, at least the same conclusions. Nonetheless, we will see in section 1.3 some technical issues about these notions.

1.2 Accepted and rejected connectives

While $\underline{\text{h}}$*armony* is meant to exclude rules that are too strong, *maximality* is meant to exclude weakening *ad hoc* of the rules, like the one needed to obtain quantum-disjunction. As a first example, I will show that the generalized rules for conjunction suit $\underline{\text{h}}$*armony*:[4]

$$\frac{\mathrm{A} \qquad \mathrm{B}}{A \wedge B}\,{\scriptstyle \wedge\mathrm{I}} \qquad \frac{A \wedge B \qquad \begin{matrix}[\mathrm{A}]\\ \vdots \\ \mathrm{D}\end{matrix}}{\mathrm{D}}\,{\scriptstyle \wedge\mathrm{gE}_1} \qquad \frac{A \wedge B \qquad \begin{matrix}[\mathrm{B}]\\ \vdots \\ \mathrm{D}\end{matrix}}{\mathrm{D}}\,{\scriptstyle \wedge\mathrm{gE}_2}$$

First, we must ask: is $A \wedge B$ the strongest conclusion we can draw from the premises of $\wedge$I? To show that it is, we must suppose that $\frac{\mathrm{A} \quad \mathrm{B}}{\mathrm{C}}\clubsuit$ holds and, using this, derive the validity of $A \wedge B \vDash C$:

$$\dfrac{A \wedge B \qquad \dfrac{A \wedge B \qquad \dfrac{[A]^1 \qquad [B]^2}{\mathrm{C}}\clubsuit}{\mathrm{C}}\,{\scriptstyle \wedge\mathrm{gE}_1,\ 1}}{\mathrm{C}}\,{\scriptstyle \wedge\mathrm{gE}_2,\ 2}$$

As is demanded by clauses (a), (b), and (c) of part (1) of $\underline{\text{h}}$*armony*, $\wedge$I is not used, but all the conditions described by this rule are exploited, and $\wedge$E is fully used.

The second part to achieve $\underline{\text{h}}$*armony* consists in showing that $A \wedge B$ is the weakest major premise that can take part in $\wedge$E. In order to obtain this, let us show that if $\frac{\mathrm{C} \qquad \begin{matrix}[\mathrm{A}]\\ \vdots \\ \mathrm{D}\end{matrix}}{\mathrm{D}}\spadesuit_1$ and $\frac{\mathrm{C} \qquad \begin{matrix}[\mathrm{B}]\\ \vdots \\ \mathrm{D}\end{matrix}}{\mathrm{D}}\spadesuit_2$ hold, $C \vDash A \wedge B$ holds as well:

[4] Read [2010], p. 565.

$$\dfrac{C \qquad \dfrac{C \qquad \dfrac{[A]^1 \qquad [B]^2}{A \wedge B}\,{\scriptstyle \wedge\mathrm{I}}}{A \wedge B}\,{\scriptstyle \spadesuit_{1},\, 1}}{A \wedge B}\,{\scriptstyle \spadesuit_{2},\, 2}$$

We can thus conclude that the generalized rules for conjunction are in $\underline{\mathrm{h}}$*armony* with each other.[5] For brevity, we will express this fact with $\underline{\mathrm{h}}(\wedge\mathrm{I},\wedge\mathrm{E})$ from now on.

As a second example, let us see that Prior's infamous rules for *tonk* do not suit $\underline{\mathrm{h}}$*armony.*[6] The rules are the usual ones:

$$\dfrac{A}{AtonkB}\,{\scriptstyle tonk\mathrm{I}} \qquad \dfrac{AtonkB}{B}\,{\scriptstyle tonk\mathrm{E}}$$

To show that $AtonkB$ is the strongest conclusion derivable from A, we must suppose that $\dfrac{\mathrm{A}}{\mathrm{C}}\,{\scriptstyle \triangle}$ and derive $AtonkB \vdash C$. However, using *tonk*E, all we can derive from $AtonkB$ is B, which we cannot use as a premise to derive C via $\triangle$. The best we can have is the derivation

$$\dfrac{\dfrac{AtonkB}{(AtonkB)tonkA}\,{\scriptstyle tonk\mathrm{I}}}{\dfrac{A}{\mathrm{C}}\,{\scriptstyle \triangle}}\,{\scriptstyle tonk\mathrm{E}}$$

But, even though this is a derivation of C from $AtonkB$, it contains an application of *tonk*I, which violates point (1.c) of the definition of $\underline{\mathrm{h}}$*armony.* The proof that $AtonkB$ is the weakest premise that enables a derivation of A has the same crucial flaw. As a conclusion, *tonk*I and *tonk*E do not suit $\underline{\mathrm{h}}$*armony.*

Let us now focus on *maximality.* Even though its formulation is very general, we will use *maximality* only to select the strongest between two alternative pairs of rules, like (ⓈI,ⓈE) and (ⓈI,ⓈE'), and never to prove that one is the strongest of all the possible pairs of rules *stricto sensu.*[7] As an example, let us see how this kind of application of *maximality* is used to exclude quantum-disjunction.

Quantum-disjunction ($\sqcup$) has the same I-rules of traditional disjunction, but the restricted E-rule

$$\dfrac{A \sqcup B \qquad \begin{matrix}[A] \\ \vdots \\ D\end{matrix} \qquad \begin{matrix}[B] \\ \vdots \\ D\end{matrix}}{D}\,{\scriptstyle \sqcup\mathrm{E}}$$

[5]Also the traditional rules for conjunction fulfill this *criterion*, but we prefer to state the result with their generalized versions, since they will be used in section 5.

[6]Tennant [201X], pp. 238,243.

[7]To do this it would be necessary to evaluate all E-rules (I-rules) in $\underline{\mathrm{h}}$*armony* with ⓈI (ⓈE).

which is applicable only when in the sub-derivations of D there are no other open assumptions apart from the ones explicitly stated (A in the left sub-derivation and B in the other one).

We know that the two disjunctions are not equivalent to each other, because distributivity of $\wedge$ over $\sqcup$ does not hold. Let us suppose we have proved both $\underline{\mathrm{h}}(\vee\mathrm{I},\vee\mathrm{E})$ and $\underline{\mathrm{h}}(\vee\mathrm{I},\sqcup\mathrm{E})$,[8] which one must be considered to be in $\underline{\mathrm{H}}$*armony*? Obviously the first, given that $\sqcup$E cannot be the strongest elimination rule in $\underline{\mathrm{h}}$*armony* with $\vee$I, since it is weaker than $\vee$E. So $\underline{\mathrm{h}}(\vee\mathrm{I},\sqcup\mathrm{E})$, but $\underline{\mathrm{H}}(\vee\mathrm{I},\vee\mathrm{E})$.

1.3 Technicalities about proposition and rule strength

In the previous section, we cheated a little, since there are some ambiguities in both the notion of proposition-strength – needed in the definition of $\underline{\mathrm{h}}$*armony* – and in that of rule-strength – needed in the definition of *maximality*.[9]

1.3.1 Proposition-strength

The *criteria* dictated by the clauses (a), (b) and (c) of the definition of $\underline{\mathrm{h}}$*armony* impose a restriction on which rules can be applied in the proof of $\underline{\mathrm{h}}(\circledS\mathrm{I},\circledS\mathrm{E})$. The only rules that are explicitly excluded are $\circledS$-rules: $\circledS$I cannot be used in the proof of point 1, and $\circledS$E cannot be used in the proof of point 2. Lacking any restriction on the rules for other connectives, the context in which the rules $\circledS$I and $\circledS$E are evaluated seems to be pivotal in the decision about their $\underline{\mathrm{h}}$*armony*. Indeed, some rules for another connective, let us say $\oslash$I and $\oslash$E, could be needed in order to prove $\underline{\mathrm{h}}(\circledS\mathrm{I},\circledS\mathrm{E})$, so that $\circledS$I and $\circledS$E are in $\underline{\mathrm{h}}$*armony* with each other only as far as they are evaluated in a system that contains $\oslash$I and $\oslash$E. As a consequence, with Tennant's definition, $\underline{\mathrm{h}}$*armony* is a global property of a pair of rules inside a system.[10]

Let us see an extreme example of this *phenomenon*: *tonk*-rules suits $\underline{\mathrm{h}}$*armony* if they are evaluated inside a system that is already trivial.[11] In the previous section, we saw that in order to prove clause (1) we needed to use both *tonk*I and *tonk*E, violating so point (1.c) of the *criterion*. In a trivial system, we can circumvent the application of *tonk*I and give the following proof of clause (1):

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[8] As is shown by the following proofs (Tennant [201X], p. 240):

$$\dfrac{A \sqcup B / A \vee B \qquad \dfrac{[A]}{D}\,\triangledown \qquad \dfrac{[B]}{D}\,\triangledown}{D}\,\sqcup/\vee\mathrm{E} \qquad\qquad \dfrac{D \qquad \dfrac{[A]}{A \sqcup B / A \vee B}\,\sqcup/\vee\mathrm{I} \qquad \dfrac{[B]}{A \sqcup B / A \vee B}\,\sqcup/\vee\mathrm{I}}{A \sqcup B / A \vee B}\,\triangle$$

[9] I thank an anonymous referee for stressing this issue.

[10] This feature - usually seen as a flaw - is shared by Prawitz's notion of harmony (Prawitz [1973]) and Belnap's notion of conservative extension (Belnap [1962]), even though not by Read's and Francez's versions of harmony (Jacinto and Read [2017] and Francez [2015]).

[11] That is, the system that we obtain by removing the *tonk*-rules enables the derivation of every sentence from every set of sentences.

$$\cfrac{\cfrac{\cfrac{AtonkB}{B}\;{\scriptstyle tonk\mathrm{E}}}{A}\;{\scriptstyle \text{Derivation in the trivial system without } tonk\text{-rules}}}{C}\;\triangle$$

Similarly, to prove clause (2), we assume $\dfrac{C}{B}\,\triangledown$ and derive $AtonkB$ from C by:

$$\cfrac{\cfrac{\cfrac{C}{B}\;\triangledown}{A}\;{\scriptstyle \text{Derivation in the trivial system without } tonk\text{-rules}}}{AtonkB}\;{\scriptstyle tonk\mathrm{I}}$$

To be completely honest, the formulations of $\underline{\text{h}}$*armony* proposed by the author in Tennant [1997] and Tennant [201X] are not equivalent with each other. Indeed, in the first formulation (that is the one we presented in section 1.1) there is an implicit veto which requires $\underline{\text{h}}$(ⓈI,ⓈE) to be provable using *only* Ⓢ-rules.[12] As a consequence, with its old characterization, $\underline{\text{h}}$*armony* is a local property of rules, that does not depend on the global system in which they are evaluated. Tennant considers his last formulation as a revised version of $\underline{\text{h}}$*armony*, but seems to be unaware of the global character he is including in it. Although the issue of whether $\underline{\text{h}}$*armony* should be a local or a global property is surely central in the evaluation of Tennant's proposal, and the author's attitude toward it is unsatisfactorily ambiguous, this problem is to some extent orthogonal to the topic of this paper, so I am happy to just gesture at this friction between the two formulations of this notion. We will see that, on the contrary, some technicalities about the notion of *rule*-strength are central to the evaluation of Steinberger's counterexample to Tennant's *criterion.*

1.3.2 Rule-strength

For the notion of rule-strength, there are two ambiguities: the first regards the comparability of rules that do not work with the same premises; the second regards the system in which the rules should be compared to each other, and is essentially connected to the ambiguity that we just saw regarding proposition-strength. Let us develop both ambiguities, starting with the first, by considering again quantum-disjunction. It follows from the restrictions imposed by $\sqcup$E on open assumptions that, for some set of premises only $\vee$E is applicable, while $\sqcup$E is not. Moreover, if we consider only the

[12] Clause (1) becomes "the conclusion of λ-introduction should be the strongest proposition that can so feature; moreover one need only appeal to λ-elimination to show this; but in so showing this, one needs to make use of all the forms of λ-elimination that are provided"; while clause (2) becomes "the major premiss for λ-elimination should be the weakest sentence that can so feature; moreover one need only appeal to λ-introduction to show this; but in so showing this, one needs to make use of all the forms of λ-introduction that are provided"; see Tennant [1997], p. 321 and Tennant [201X], p. 239.

subset of premises for which *both* rules are applicable, $\vee$E and $\sqcup$E seem to have the same strength, allowing the derivation of the same conclusion from the same assumptions. So there seem to be three ways of confronting these rules:[13]

1. They are not comparable, since there have different sets of possible premises;

2. They are comparable, because they can be applied to the same premises, and since in these cases they derive the same conclusions, they have the same strength;

3. They are comparable, since the derivations that one ($\sqcup$E) enables are a subset of the derivations that the other one ($\vee$E) enables, and so the second rule ($\vee$E) is at least as strong as the first ($\sqcup$E).

In section 1.2, we tacitly assumed the third alternative, concluding that $\vee$E is stronger than $\sqcup$E and so the only rule that suits *maximality* with respect to $\vee$I. Someone could raise the issue of whether this is the right choice for a harmony *criterion*. I will briefly argue that it is both a good choice in general and one that is faithful to Tennant's approach. To see that it is the best alternative, just consider the reason why *maximality* was introduced in the first place. Tennant himself presents it as a way to find a *single*, univocal E-rule for each I-rule, and *vice-versa*.[14] Nonetheless, alternatives 1 and 2 give no way to choose between $\sqcup$E and $\vee$E giving hence no answer to our request for univocity. By contrast, the third alternative discards $\sqcup$E in favor of $\vee$E.[15] Tennant himself, even though without considering explicitly $\sqcup$E as an alternative, asserts that the standard rules for disjunctions are selected by $\underline{\text{H}}$*armony*.[16] Hence, the third reading of "rule-strength" is the only acceptable, both for disjunction and in general.

In the previous paragraph, we just assumed that without restrictions on the comparability of rules, $\vee$E turns out stronger than $\sqcup$E, but to be precise all we can prove is that the first is at least as strong as the second. To prove that $\vee$E is stronger than $\sqcup$E, we need to display a logical consequence that can be derived using $\vee$E but that cannot be derived using $\sqcup$E. In the previous section, we just claimed that distributivity of $\wedge$ over $\sqcup$ does not hold, while distributivity of $\wedge$ over $\vee$ holds, assuming that this was enough to settle the issue. However, it should be remarked that distributivity of $\wedge$ over $\sqcup$ *does* hold in some systems. As an example, it holds in a system with the usual rules for $\supset$. Moreover, in those systems $\sqcup$E and $\vee$E enables

[13] I thank an anonymous referee for stressing this issue.

[14] Apparently following a suggestion from Peter Schroeder-Heister; see Tennant [1987], pp. 94-95.

[15] Or at least, it does so if we settle another ambiguity in the right way, as I will explain in the rest of this section.

[16] Tennant [201X], p. 240.

precisely the same derivations, since the second becomes derivable from the first:[17]

$$
\dfrac{\Gamma \qquad \dfrac{\Delta \qquad \dfrac{A \sqcup B \qquad \dfrac{\dfrac{\begin{matrix}[\Gamma]^1 \quad [A]^3 \\ \vdots \\ C\end{matrix}}{\Gamma^{\supset} \supset C}\,{\scriptstyle \supset \mathrm{I}_1}}{\Delta^{\supset} \supset \Gamma^{\supset} \supset C}\,{\scriptstyle \supset \mathrm{I}} \qquad \dfrac{\dfrac{\begin{matrix}[\Delta]^2 \quad [B]^3 \\ \vdots \\ C\end{matrix}}{\Gamma^{\supset} \supset C}\,{\scriptstyle \supset \mathrm{I}}}{\Delta^{\supset} \supset \Gamma^{\supset} \supset C}\,{\scriptstyle \supset \mathrm{I}_2}}{\Delta^{\supset} \supset \Gamma^{\supset} \supset C}\,{\scriptstyle \sqcup \mathrm{E}_3}}{\Gamma^{\supset} \supset C}\,{\scriptstyle \supset \mathrm{E}}}{C}\,{\scriptstyle \supset \mathrm{E}}
$$

The lesson that we should learn is that the strength of a rule depends on the system in which it is evaluated. More specifically, a rule can be stronger than another in a given system, even though they are equivalent to each other in a different system, so the verdict of the *criterion* of *maximality* depends on the system under scrutiny. Put in other words, *maximality* is a global principle, not a local one.

Someone could argue that locality can be preserved by evaluating the strength of a rule - and so *maximality* - in a system composed of only the *harmonious* rules under scrutiny.[18] Let us take as an example the rules:[19]

$$
\dfrac{\begin{matrix}[A] \\ \vdots \\ B\end{matrix}}{A \supset B}\,{\scriptstyle \supset \mathrm{I}} \qquad \dfrac{B}{A \supset B}\,{\scriptstyle \mathrm{Weak}\supset \mathrm{I}} \qquad \dfrac{A \supset B \qquad A}{B}\,{\scriptstyle \supset \mathrm{E}}
$$

We can argue that $\supset$I is stronger than Weak$\supset$I and we can do it *locally*. Indeed, from the absence of rules that discharge assumptions in a system composed only by Weak$\supset$I and $\supset$E, it follows that Weak$\supset$I cannot derive $A \supset A$ from an empty set of assumptions. On the contrary, $A \supset A$ is provable by $\supset$I in a system composed only by $\supset$I and $\supset$E. Hence, in this case, there is hope for a local characterization of rule-strength and *maximality*.

Unfortunately, this situation does not seem to be generalizable. Indeed, let us focus again on $\vee$E and $\sqcup$E. The only difference between these two rules

[17] I write $\Gamma^{\supset}$ for the conditional $\gamma_1 \supset \ldots \supset \gamma_n$, such that $\forall_{1 \leq i \leq n} \gamma_i \in \Gamma$, and $\dfrac{\Gamma \qquad \Gamma^{\supset} \supset C}{C}\,{\scriptstyle \supset \mathrm{E}}$ for as many applications of $\dfrac{\gamma_i \qquad \Gamma \setminus \{\gamma_{1 \leq j < i}\}^{\supset} \supset C}{\Gamma \setminus \{\gamma_{1 \leq j \leq i}\}^{\supset} \supset C}\,{\scriptstyle \supset \mathrm{E}}$ as there are formulae γ_i in Γ.

[18] This is essentially the move we described in the previous section regarding proposition-strength and Tennant definition of *harmony* in Tennant [1997].

[19] Rule Weak$\supset$I is investigated in Milne [2010].

is that $\vee$E can be applied to derivations $\Gamma, A \vdash C$ and $\Delta, B \vdash C$[20] in which more assumptions are used to derive a conclusion, while $\sqcup$E cannot. Of course, this difference between the two rules is interesting only in a system in which derivations can have more than one open assumption, since in a system in which derivations can have only one open assumption $\sqcup$E can be applied all the times in which $\vee$E can. Nonetheless, it is easy to observe that in a system composed only by $\vee$I and $\vee$E we start with a single assumption and we gain new open assumptions only when $\vee$E discharges vacuously A or B, or when it intentionally avoids discharging A or B even though they occur as open assumptions in the respective subderivations. In both these cases, it is evident that we can find a derivation of the same conclusion from just one of the open assumptions of the original derivation. So we do not really need more than one open assumption in a system composed only by $\vee$I and $\vee$E, at least if we assume monotonicity of the derivability relation – that is, if $\Gamma \vdash C$ then $\Gamma, A \vdash C$. So, if monotonicity is assumed $\vee$E and $\sqcup$E have the same strength in the system composed respectively by $\vee$I and $\vee$E, and by $\vee$I and $\sqcup$E. That is, according to the *local* definition of this notion, $\vee$E and $\sqcup$E seem to have the same strength, and as a consequence *maximality* is unable to select just one of them. As far as I know, Tennant never discusses these ambiguities in the notion of rule-strength, and what we have said here cannot set fully this issue. Luckily enough, for all the rules that we will evaluate in this paper, starting with Steinberger's quantifier, the issue of *maximality* can be faced locally.

1.4 Justification of the restrictions

Since our reply to Steinberger's alleged counterexample heavily relies on the extra requirements (a), (b) and (c) of $\underline{\mathrm{h}}$*armony*, we should argue for their *prima facie* pertinence.[21]

The requirement of $\underline{\mathrm{H}}$*armony* asks for a perfect affinity between I and E-rules and between the occurrences of Ⓢ-formulae in them, by imposing: that the meaning given to Ⓢ by its I-rule (E-rule) justifies the occurrence of AⓈB as a major premise (conclusion) in ⓈE (ⓈI); that the entire meaning defined by the justifying rule and nothing more is used in the justification procedure; that there are no weaker (stronger) formulae that can occur as major premise (conclusion) of ⓈE (ⓈI); that there are no stronger rules to pair with ⓈI or ⓈE that suit the previous requirements.

Although they could seem puzzling at first sight, all these requirements are standard in the most common antirealist theories of meaning, where it is usual to distinguish between:

[20] Together with the premise $A \vee B$, of course.

[21] I thank an anonymous referee who reviewed an earlier draft of this paper for stressing the point of the apparent *ad hocness* of these requirements.

- A formal *criterion* corresponding to the justification of E-rules (and eventually I-rules) from the I-rules (E-rules);
- A formal *criterion* corresponding to the requirement of full usage of the justifying rules, used to select the strongest pair of rules which suits the first point.

The term '*harmony*' is usually reserved for the first *criterion*, while the second (or the union of the two) is labeled '*stability*'.[22]

In the usual picture, harmony is characterized using the availability of reduction procedures and not imposing conditions on the strength of the propositions: a pair of I and E-rules is harmonious iff in every derivation, all *maximal formulae* – that is sentences that are conclusions of I-rules and major premises of E-rules – can be removed via a reduction procedure.[23] Where conditions on the strength of the rules become relevant is with stability, which is developed to select the strongest pair of rules that suits harmony. The usual example of an unstable rule is quantum-disjunction, which we already saw, and the *criterion* for stability asks that ⓈE fully uses the meaning given to Ⓢ by ⓈI, that is the conditions under which AⓈB can be derived according to ⓈI.[24]

There seems to be a parallelism between Tennant's account and the more traditional account based on reduction procedures.[25] Indeed, $\underline{\text{h}}$*armony*, like harmony, is used to reject rules that are too strong – *tonk*-rules *in primis* –, and *maximality*, like stability, is used to exclude rules that are too weak – quantum-disjunction *in primis*. Moreover, clause (c) is devised explicitly to exclude *maximal formulae* in the derivations that show $\underline{\text{h}}$*armony*.[26]

Hence, Tennant's *criterion* of $\underline{\text{H}}$*armony* is not *prima facie* unconvincing, or at least not much more unconvincing than the other harmony *criteria* in the literature. What is peculiar in Tennant's account of harmony and stability, or better $\underline{\text{h}}$*armony* and $\underline{\text{H}}$*armony*, is that the requirement of full use is not part of the second notion, but of the first, occurring explicitly in clauses (a) and (b). *Vice-versa*, *maximality*, which is used to obtain $\underline{\text{H}}$*armony* (that is stability), does not make any explicit reference to the notion of full use. So Tennant tries to sever full use from *maximality* and rule-strength, relegating

[22]Dummett [1991] is the source of these two labels.

[23]There are some technicalities involved here, and some disagreements about whether the availability of single passes of reduction is enough for harmony (Francez [2015]), or a terminating procedure is instead needed (Prawitz [1973]).

[24]See Jacinto and Read [2017] (p. 373) and Tranchini [2016] (pp. 17-18) for some technical developments of this notion.

[25]Tennant insists that they are equivalent to each other, and equivalent to a third account based on conservative extension (Tennant [201X], p. 233), even though Steinberger gives good reasons to doubt this conclusion; Steinberger [2013]. Even though part of Steinberger's argument could be undermined by our considerations in section 5, its conclusion remains valid nonetheless, as I hope will be clear.

[26]See the proof that *tonk*-rules are not $\underline{\text{h}}$*armonious* in section 1.2.

the first notion inside $\underline{\text{h}}$*armony* and giving a characterization of *maximality* that does not deal with full use but only with rule-strength.

We will see at the end of this paper that such an option leads to unacceptable conclusions beyond the rigid corners of propositional logic, and that this is the lesson that we should learn from Steinberger's counterexample.[27] Moreover, we will show that reconnecting full use with rule-strength, a move that common sense seems to encourage, leads $\underline{\text{H}}$*armony* to even worse problems.

2 Steinberger's objection

After the display of Tennant's *criterion* and one of its 'virtuous' applications, let us now consider the supposed counterexample suggested by Steinberger, which questions the capacity of $\underline{\text{h}}$armony to exclude E-rules that are too strong. If this counterexample to $\underline{\text{h}}$armony is successful, the second part of the definition of $\underline{\text{H}}$*armony* – that is *maximality* – will not block it, recommending the adoption of the *strongest* rules. [28]

Steinberger displays the following rules for a 'strengthened' existential quantifier:

$$\dfrac{A(t)}{\underline{\dashv} x A(x)}\ {\scriptstyle \underline{\dashv}\mathrm{I}} \qquad\qquad \dfrac{\underline{\dashv} x A(x) \qquad \begin{array}{c}[A(t)]\\ \vdots \\ \mathrm{D}\end{array}}{\mathrm{D}}\ {\scriptstyle \underline{\dashv}\mathrm{E}}$$

where no restriction is imposed for the applicability of $\underline{\dashv}$E. That is, the term t can occur in every assumption on which D depends in the right sub-derivation (not only $A(t)$), in D and in $\underline{\dashv} x A(x)$. More briefly, $\underline{\dashv}$I is identical to $\exists$I, while $\underline{\dashv}$E differs from $\exists$E only because the first lacks any restriction in the occurrence of terms in sub-derivations.

This pair of rules is dangerous, since it makes it possible to prove that every propositional function true of *some* object is true of *every* object. Indeed, we can easily derive $A(a) \vDash A(b)$, for every pair of terms a and b:

$$\dfrac{\dfrac{A(a)}{\underline{\dashv} x A(x)}\ {\scriptstyle \underline{\dashv}\mathrm{I}} \qquad [\mathrm{A(b)}]^1}{\mathrm{A(b)}}\ {\scriptstyle \underline{\dashv}\mathrm{E},\ 1}$$

[27] This, or the need to accept an infinitary formulation of the rules for the quantifiers.

[28] The objection appeared for the first time in Steinberger [2008], and later in Steinberger [2009] and Steinberger [2013], without substantial changes.

And since we can use any term b, we can choose one that is suitable for a subsequent application of $\forall$I, so to conclude $A(a) \vDash \forall x A(x)$. This is enough to show that a good harmony *criterion* should admit $\exists$-rules, but reject $\Finv$-rules. However, Steinberger maintains that both the rules for $\exists$ and those for $\Finv$ are in $\underline{\mathrm{h}}$*armony*, and thus for *maximality* $\Finv$ is the only constant characterized by rules in $\underline{\mathrm{H}}$*armony*, it being stronger than the existential quantifier.[29] So, to endorse $\underline{\mathrm{H}}$*armony* would impose to accept $\Finv$, and reject $\exists$.

Let us see the details of Steinberger's proof. In order to prove $\underline{\mathrm{h}}(\exists\mathrm{I},\exists\mathrm{E})$ and $\underline{\mathrm{h}}(\Finv\mathrm{I},\Finv\mathrm{E})$, first of all we show that $\Finv x A(x)$ and $\exists x A(x)$ are the strongest conclusions that can be derived by the premise of I-rule (common to each constant). Assuming $\dfrac{\mathrm{A(t)}}{\mathrm{C}}\,\star$, we prove $\Finv x A(x) \vDash C$ and $\exists x A(x) \vDash C$:

$$\dfrac{\exists x A(x) \qquad \dfrac{[\mathrm{A(a)}]^1}{\mathrm{C}}\,\star}{\mathrm{C}}\;{\scriptstyle \exists\mathrm{E},\,1}$$

C following from $A(t)$ for every t, we choose a term a occurring neither in C nor in $\exists x A(x)$, to make possible the application of $\exists$E. The same proof holds trivially for $\Finv$E, and in this case there is no need to restrict the choice of term a.

Now we prove that $\Finv x A(x)$ and $\exists x A(x)$ are the weakest major premises that can take part in the respective E-rule, by considering the case for $\exists$ and generalizing the result to $\Finv$. Assume $\dfrac{\mathrm{C} \qquad \begin{array}{c}[\mathrm{A(a)}]\\ \vdots \\ \mathrm{D}\end{array}}{\mathrm{D}}\,\blacktriangle$, with the restriction that a does not occur in any other assumptions on which D depends, in D or in C. We can show that $C \vDash \exists x A(x)$ holds, in this way:

$$\dfrac{\mathrm{C} \qquad \dfrac{[\mathrm{A(a)}]^1}{\exists x A(x)}\;{\scriptstyle \exists\mathrm{I}}}{\exists x A(x)}\;{\scriptstyle \blacktriangle,\,1}$$

Here we have used a term a which follows the restrictions imposed by the application of $\blacktriangle$. We can prove $C \vDash \Finv x A(x)$ in the same way, using $\Finv$I and neglecting the restrictions on the occurrence of terms.

We have $\underline{\mathrm{h}}(\exists\mathrm{I},\exists\mathrm{E})$ and $\underline{\mathrm{h}}(\Finv\mathrm{I},\Finv\mathrm{E})$, so we can conclude, the two pairs sharing the same I-rule[30] and $\Finv$E being stronger than $\exists$E, that $\underline{\mathrm{H}}(\Finv\mathrm{I},\Finv\mathrm{E})$ holds but $\underline{\mathrm{H}}(\exists\mathrm{I},\exists\mathrm{E})$ does not.

[29]Given $A(a) \vDash_{\Finv} A(b)$ and $A(a) \nvDash_{\exists} A(b)$.

[30]$\exists$I is identical to $\Finv$I.

3 Tennant's answer and its problems

To answer Steinberger's objection, Tennant analyses the counterexample in a sequent setting, where the rules for $\exists$ and $\rotatebox[origin=c]{180}{F}$ become:[31]

$$\frac{\Gamma \vdash A(t)}{\Gamma \vdash \exists x A(x)}\ {\scriptstyle \text{R-}\exists} \qquad \frac{\Gamma, A(a) \vdash D}{\Gamma, \exists x A(x) \vdash D}\ {\scriptstyle \text{L-}\exists}$$

$$\frac{\Gamma \vdash A(t)}{\Gamma \vdash \rotatebox[origin=c]{180}{F} x A(x)}\ {\scriptstyle \text{R-}\rotatebox[origin=c]{180}{F}} \qquad \frac{\Gamma, A(t) \vdash D}{\Gamma, \rotatebox[origin=c]{180}{F} x A(x) \vdash D}\ {\scriptstyle \text{L-}\rotatebox[origin=c]{180}{F}}$$

With the restriction $a \notin \Gamma \cup \{D\}$ for the applicability of L-$\exists$.

With this reformulation, it is still possible to prove that $\exists x A(x)$ ($\rotatebox[origin=c]{180}{F} x A(x)$) is the strongest right-conclusion possible for R-$\exists$ (R-$\rotatebox[origin=c]{180}{F}$). Indeed, the derivation

$$\cfrac{\cfrac{\cfrac{A(a) \vdash A(a)}{A(a) \vdash C}\ \star \qquad \Gamma, C \vdash D}{\Gamma, A(a) \vdash D}\ {\scriptstyle \text{Cut}}}{\Gamma, \exists x A(x) \vdash D}\ {\scriptstyle \text{L-}\exists}$$

(with a suitable for L-$\exists$) shows that everything that follows from an arbitrary conclusion C of R-$\exists$, follows also from $\exists x A(x)$. Of course, we can adapt the demonstration to $\rotatebox[origin=c]{180}{F}$ in the usual way. It is also possible to prove that $\exists x A(x)$ ($\rotatebox[origin=c]{180}{F} x A(x)$) is the weakest left-conclusion possible for for L-$\exists$ (L-$\rotatebox[origin=c]{180}{F}$).[32]

Nonetheless, by the use of the Cut rule, sequent calculus makes explicit that the transitivity of logical consequence is needed to prove $\underline{\text{h}}$(L-$\exists$,R-$\exists$) and $\underline{\text{h}}$(L-$\rotatebox[origin=c]{180}{F}$,R-$\rotatebox[origin=c]{180}{F}$). As a consequence, since Tennant notoriously rejects the transitivity of logical consequence,[33] and the formal counterpart of this refusal is that the Cut is acceptable only in those systems in which it is admissible,[34] the proofs of $\underline{\text{h}}$*armony* just displayed are acceptable only if the rule of Cut is admissible in the system in which they are formulated.

This additional requirement excludes Steinberger's counterexample, the Cut in

$$\cfrac{\cfrac{A(a) \vdash A(a)}{A(a) \vdash \rotatebox[origin=c]{180}{F} x A(x)}\ {\scriptstyle \text{R-}\rotatebox[origin=c]{180}{F}} \qquad \cfrac{A(b) \vdash A(b)}{\rotatebox[origin=c]{180}{F} x A(x) \vdash A(b)}\ {\scriptstyle \text{L-}\rotatebox[origin=c]{180}{F}}}{A(a) \vdash A(b)}\ {\scriptstyle \text{Cut}}$$

[31] Tennant [2010]

[32] In this reformulation, right-conclusion and left-conclusion replace conclusion and major premise, respectively.

[33] The reasons for this choice are exposed thoroughly in Tennant [1987]

[34] Tennant [2010], pg. 465.

not being admissible (while Cut is admissible in the system with $\exists$ in place of Ǝ).

Anyway, Steinberger has correctly pointed out that this addition makes the preceding $\underline{\text{h}}$*armony criterion* completely useless.[35] Indeed, all the work for excluding those E-rules that are too strong is done by the admissibility of Cut, so that persisting in assuming $\underline{\text{h}}$*armony* in combination with it is completely pointless.[36] Moreover, admissibility of Cut is equivalent to normalizability, which is the main ingredient of Prawitz's *harmony criterion*,[37] so that Tennant's *criterion* ends up loosing its autonomy. In conclusion, what Tennant is proposing is not a reply to Steinberger's counterexample to $\underline{\text{H}}$*armony*, on the contrary it is to change completely both the *criterion* and the system in which it is applied.[38]

4 Wright's objection

In this section, we will see an objection to $\underline{\text{H}}$*armony* raised by Crispin Wright. In the next section, I will argue that Tennant's answer to this objection can be generalized to solve Steinberger's counterexample as well.[39] Then, in section 6, I will argue that there are stronger and apparently unsolvable objections to Tennant's *criterion*, highlighted by his answer to Wright.

Let us consider a *tonk*-like connective $\heartsuit$ with I-rules identical to those for $\vee$ and E-rules identical to those for $\wedge$:

$$\frac{\mathrm{A}}{A\heartsuit B}\,{\scriptstyle \heartsuit \mathrm{I}_1} \qquad\qquad \frac{\mathrm{B}}{A\heartsuit B}\,{\scriptstyle \heartsuit \mathrm{I}_2}$$

$$\frac{A\heartsuit B}{\mathrm{A}}\,{\scriptstyle \heartsuit \mathrm{E}_1} \qquad\qquad \frac{A\heartsuit B}{\mathrm{B}}\,{\scriptstyle \heartsuit \mathrm{E}_2}$$

To prove that $A\heartsuit B$ is the strongest conclusion derivable both from A and from B – clause (1) –, let us assume $\frac{\mathrm{A}}{\mathrm{C}}\,{\scriptstyle \vartriangle_1}$ and $\frac{\mathrm{B}}{\mathrm{C}}\,{\scriptstyle \vartriangle_2}$ and prove $A\heartsuit B \vDash C$ by one of these derivations:

$$\cfrac{\cfrac{A\heartsuit B}{\mathrm{A}}\,{\scriptstyle \heartsuit \mathrm{E}_1}}{\mathrm{C}}\,{\scriptstyle \vartriangle_1} \qquad\qquad \cfrac{\cfrac{A\heartsuit B}{\mathrm{B}}\,{\scriptstyle \heartsuit \mathrm{E}_2}}{\mathrm{C}}\,{\scriptstyle \vartriangle_2}$$

[35] Steinberger [2011].

[36] Steinberger [2008].

[37] Prawitz [1973].

[38] This is radically different from Tennant's *desideratum* expressed in Tennant [1987] (chapter 23, especially pp. 253-265) and in Tennant [1997] (chapter 10), that is to prove that the *criteria* lead to coextensive systems in sequent calculus and in natural deduction.

[39] Both Wright's objection and Tennant's solution are in Tennant [201X], but the application to Steinberger's objection is mine.

To prove that $A \heartsuit B$ is the weakest premise from which we can derive both A and B – clause (2) –, let us assume $\dfrac{\text{C}}{\text{A}}\triangledown_1$ and $\dfrac{\text{C}}{\text{B}}\triangledown_2$ and prove $C \vDash A \heartsuit B$ by one of these derivations:

$$\dfrac{\dfrac{\text{C}}{\text{A}}\,\triangledown_1}{A \heartsuit B}\,\heartsuit\text{I}_1 \qquad\qquad \dfrac{\dfrac{\text{C}}{\text{B}}\,\triangledown_2}{A \heartsuit B}\,\heartsuit\text{I}_2$$

So, it seems that $\underline{\text{h}}(\heartsuit\text{I},\heartsuit\text{E})$.

It is impossible to exclude this pair of sets of rules by the *maximality* clause of $\underline{\text{H}}$*armony*. Indeed if $\underline{\text{h}}(\heartsuit\text{I},\heartsuit\text{E})$ holds, $\heartsuit$I is equivalent to $\vee$I and $\heartsuit$E is equivalent to $\wedge$E:

- Given $\heartsuit$I ($\vee$I), $\heartsuit$E (not $\vee$E) is the strongest E-rule in $\underline{\text{h}}$*armony* with it;
- Given $\heartsuit$E ($\wedge$E), $\heartsuit$I (not $\wedge$I) is the strongest I-rule in $\underline{\text{h}}$*armony* with it.[40]

So not only *maximality* does not exclude $\heartsuit$, but it even excludes $\vee$ or $\wedge$.

If this were all the story, Tennant's *criterion* would be useless against trivial paradoxical connectives like $\heartsuit$. Nevertheless, Tennant points out that in the definition of $\underline{\text{h}}$*armony*, in addition to the clauses about the strength of premises and conclusions, we have some restrictions on how these properties can be proved:

- to prove that the conclusion of the I-rule is the strongest possible (1), one needs: to exploit *all the conditions* described by the I-rule (1a); and to make *full use* of the E-rule (1b);
- to prove that the major premise of the E-rule is the weakest possible (2), one needs: to exploit *all the conditions* described by the E-rule (2a); and to make *full use* of the I-rule (2b).

Tennant never express explicitly what he means with "full use or a rule", but from his reply to Wright we can infer that he tacitly assumes the following:

Criterion 4.1 (Full use (Tennant))**.** *ⓈE (ⓈI) is fully used in the proof of (1) (in the proof of (2)) iff all the rules belonging to ⓈE (ⓈI) are applied in the proof. All the conditions described by ⓈI (ⓈE) are exploited in the proof of (1) (in the proof of (2)) iff, for each rule in ⓈI (ⓈE), the result of replacing its conclusion (major premise) with a new sentence occurs in the proof.*

[40] In both cases the result is a consequence of the paradoxical nature of $\heartsuit$.

The proofs of $\underline{\text{h}}$*armony* for $A \heartsuit B$ breach these requirements. For example, to prove (1), we can use $\heartsuit$E$_1$ and $\triangle_1$, or $\heartsuit$E$_2$ and $\triangle_2$, but we can use neither both the E-rules nor both the contexts given by the I-rules.[41] So we violate both (1a) and (1b). Symmetrically, to prove (2), we can use only one of the I-rules for $\heartsuit$ and only one of the $\triangledown$-rules, so that we violate both (2a) and (2b). As a conclusion, interpreting the ask of full use as suggested in *criterion* 4.1, $\underline{\text{h}}$*armony* excludes Wright's counterexample.

5 Steinberger again

The main point of my reply to Steinberger is that his counterexample, like Wright's one, overlooks the clauses (a) and (b) of the definition of $\underline{\text{h}}$*armony*. This flaw in his argument is not immediately evident just because the request for full use is a little tricky for the rules for quantifiers. Indeed, Tennant's interpretation of this request via *criterion* 4.1 is evidently best suited for the propositional case, and its application to the rules for quantifiers seems improper. Admittedly, $\exists$I and $\exists$E suit the *criterion* 4.1, if this is taken at face value, but there are reasons to call in question this approach. Indeed, the terms occur schematically in $\exists$I and $\exists$E, and Steinberger does not discuss substitution explicitly. When substitution is taken into account, it is natural to treat each of these rules as a recipe to build a (possibly infinite) set of rules: one for each term. This generality raises the problem of how such a recipe can be used fully, and Tennant's interpretation of full use seems to offer no solution to this problem.

As an example, while in its explicit formulation $\exists$I enables the derivation of $\exists x A(x)$ only from $A(t)$, since both A and t occur schematically in it, the rule enables the construction of inferences from every saturation of an open formula $B(\ )$ with a term to the conclusion $\exists x B(x)$. We are not interested in the substitution of open formulae, since it works like the substitution of sentences in the propositional case, but the substitution of terms is a key ingredient of the rules for quantifiers. The schematicity of $\exists$I regarding its terms gives a recipe to build a (possibly infinite) multitude of introductions for the same conclusion, just like $\vee$I has two introductions of the same conclusion. We could make explicit this generality of $\exists$I in the following way:

$$\frac{A(t)}{\exists x A(x)}\ {\scriptstyle \exists \mathrm{I}} \quad \text{for every term } t \qquad\qquad \frac{\exists x A(x) \qquad \begin{matrix}[\mathrm{A(t)}]\\ \vdots \\ \mathrm{D}\end{matrix}}{\mathrm{D}}\ {\scriptstyle \exists \mathrm{E}} \quad \text{for every term } t$$

[41] That is, both $\triangle_1$ and $\triangle_2$.

The same analysis holds for $\exists$I and $\Finv$E as well, since there are no restrictions on the terms that occur in them. We will see that $\exists$E has to be treated in a quite different way.

In propositional logic, we interpreted the request of full use in clauses (a) and (b) of $\underline{\mathrm{h}}$*armony* as asking for the application of all the available I and E-rules for the connective at issue. In the proof of the first point of $\underline{\mathrm{h}}$($\vee$I,$\vee$E), $\vee$I is fully used because both its rules are applied, while in the proof of the first point of $\underline{\mathrm{h}}$(♡I,♡E) ♡I is not fully used because only one of its rules is applied. Nevertheless, while this interpretation of the "full use" requirement in the field of propositional logic seems acceptable, it cannot be extended directly to $\Finv$I, $\exists$I and $\Finv$E, let alone $\exists$E, lacking any treatment for schematicity.

In the remaining part of this section, we will see two ways of interpreting the full use clause for the quantificational rules:

An infinitary reading that dismantles Steinberger's counterexample and accepts an infinitary reformulation of the standard rules for the quantifiers;

A finitary reading that dismantles Steinberger's counterexample but is unable to save the standard rules for the quantifiers.

The first approach maintains *criterion* 4.1 for full use, by connecting quantifiers to (possibly) infinitary propositional connectives. The second approach maintains the traditional finitary formulation of the quantifiers and updates the *criterion* of full use proposed by Tennant.

By dropping the finitary restrictions on the shape and numbers of the rules, we could interpret $\Finv$-rules not just as finite recipes for forming an infinite number of inferences, but as an infinite number of rules *simpliciter*.[42] We will label with ∞ the infinitary versions of the rules for $\Finv$ and $\exists$, and index with numbers the rules that belong to them:[43]

[42] An anonymous referee asked why the ranging of $\Finv$I and $\Finv$E over term variables produces an infinite number of rules, while the ranging of $\wedge$I and $\wedge$E over propositional variables does not. The answer is simple: as we already noticed, if we vary t in the premise of $\Finv$I the conclusion remains the same, while if we change a propositional variable in a premise of $\wedge$I also the conclusion changes.

[43] Note that we use the same number to index the rule and the term which occurs in the rule.

$$\dfrac{A(t_1)}{\exists x A(x)}\,{\scriptstyle \exists \mathrm{I}_1} \qquad \cdots \qquad \dfrac{A(t_i)}{\exists x A(x)}\,{\scriptstyle \exists \mathrm{I}_i} \qquad \cdots \qquad \cdots$$

$$\dfrac{\exists x A(x) \qquad \begin{matrix}[A(t_1)]\\ \vdots \\ \mathrm{D}\end{matrix}}{\mathrm{D}}\,{\scriptstyle \exists \mathrm{E}_1} \qquad \cdots \qquad \dfrac{\exists x A(x) \qquad \begin{matrix}[A(t_i)]\\ \vdots \\ \mathrm{D}\end{matrix}}{\mathrm{D}}\,{\scriptstyle \exists \mathrm{E}_i} \qquad \cdots \qquad \cdots$$

Using this manoeuvre, we can read the full use clause of $\underline{\mathrm{h}}$*armony* at face value, as we did for the propositional case. Someone could observe that this reformulation is not infinitary after all. Indeed all the rules that belong to $\exists \mathrm{I}_\infty$ have a finitary structure, they are just infinite in numbers. Nonetheless, in order to use $\exists \mathrm{I}_\infty$ fully, we need to apply all the rules $\exists \mathrm{I}_n$ in the proofs of (1) and (2), so obtaining an infinitary proof *stricto sensu*.

Unfortunately for Steinberger, it seems impossible to use all the rules belonging to $\exists \mathrm{I}_\infty$ and to $\exists \mathrm{E}_\infty$ in the proof of $\underline{\mathrm{h}}(\exists \mathrm{I}_\infty, \exists \mathrm{E}_\infty)$. Indeed, the proofs that were given by Steinberger for the clauses (1) and (2) of $\underline{\mathrm{h}}(\exists \mathrm{I}, \exists \mathrm{E})$ can be easily adapted for the infinitary framework: we just need to add the appropriate label to the rules. So, to prove part (1) of $\underline{\mathrm{h}}(\exists \mathrm{I}_\infty, \exists \mathrm{E}_\infty)$ we use just one I-rule – let us say, $\exists \mathrm{I}_i$ – and we exploit all the conditions described by just the correlate $\exists \mathrm{E}_i$. In the same way, to prove part (2) of $\underline{\mathrm{h}}(\exists \mathrm{I}_\infty, \exists \mathrm{E}_\infty)$ we use just one E-rule – let us say, $\exists \mathrm{E}_i$ – and we exploit all the conditions described by just the correlate $\exists \mathrm{I}_i$. We can indeed choose among all $i \in \mathbb{N}$, but we can pick only one number, and so only one pair of rules. As a conclusion, there is no way of satisfying the interpretation given by *criterion* 4.1 of (a) and (b), and $\underline{\mathrm{h}}(\exists \mathrm{I}_\infty, \exists \mathrm{E}_\infty)$ does not hold.[44] The best we can obtain is an infinitary derivation like

$$\dfrac{\exists x A(x) \text{ (or C)} \qquad \begin{matrix}\dfrac{\dfrac{\exists x A(x) \text{ (or C)} \qquad \dfrac{[A(t_i)]^i}{\text{C (or } \exists x A(x))}\,{\scriptstyle \bigstar_i \text{ (or } \exists \mathrm{I}_i)}}{\text{C (or } \exists x A(x))}\,{\scriptstyle \exists \mathrm{E}_1 \text{ (or } \blacktriangle_1)}}{} \\ \vdots \\ \text{C (or } \exists x A(x))\end{matrix}}{\text{C (or } \exists x A(x))}\,{\scriptstyle \exists \mathrm{E}_i \text{ (or } \blacktriangle_i), \text{ i}}$$

$$\vdots$$

in which only one I-rule (or the conditions described by it) is applied and only one E-rule (or the conditions described by it) has an active role in the

[44] Moreover, in order to have $\underline{\mathrm{h}}$*armony* the full use of ($\exists \mathrm{E}_\infty$) $\exists \mathrm{I}_\infty$ and of the conditions described by $\exists \mathrm{E}_\infty$ ($\exists \mathrm{I}_\infty$) should not only be possible, but even necessary.

derivation, since all the other E-rules make no change in the derivation. In this case $\bigstar_i$ and $\dashv\mathrm{E}_i$ for clause (1), and $\dashv\mathrm{I}_i$ and $\blacktriangle_i$ for clause (2).

The reason why there cannot be full use of $\dashv$I and $\dashv$E is that – like for Crispin Wright's counterexample –, while the first has the same structure of $\vee$I, the second has the same structure of $\wedge$gE. Indeed, to prove $\underline{\mathrm{h}}(\vee\mathrm{I},\vee\mathrm{E})$, we make full use of $\vee$I (or of the conditions described by it) because $\vee$E asks for two sub-derivations (one for each I-rule). But, since each of $\dashv\mathrm{E}_\infty$ asks for just one sub-derivation, it is impossible to make full use of all the rules in $\dashv\mathrm{I}_\infty$ in the same way. On the other side, to prove $\underline{\mathrm{h}}(\wedge\mathrm{I},\wedge\mathrm{E})$ we need to use both $\wedge\mathrm{gE}_1$ and $\wedge\mathrm{gE}_2$ (or the conditions described by them), since $\wedge$I has two premises and each $\wedge$gE discharges only one assumption. So, in this derivation we make full use of $\wedge$I and $\wedge$E, and it is quite obvious that we cannot obtain the same result in other ways. Following this train of thought, in order to make full use of $\dashv\mathrm{E}_\infty$ there should be just one I-rule for $\dashv$, with as many premises as the rules in $\dashv\mathrm{E}_\infty$.[45] But, since each $\dashv$I has only one premise, we only need one $\dashv\mathrm{E}_i$ (the right one) to discharge it.

If we remain in the infinitary framework, we can see that the rules for quantifiers suit *criterion* 4.1. Indeed let us consider the following formulation of the elimination rule for $\exists$:[46]

$$\dfrac{\exists x A(x) \qquad \begin{matrix}[A(t_1)]\\ \vdots \\ \mathrm{D}\end{matrix} \qquad \begin{matrix}\dots\\ \vdots \\ \dots\end{matrix} \qquad \begin{matrix}[A(t_i)]\\ \vdots \\ \mathrm{D}\end{matrix} \qquad \begin{matrix}\dots\\ \vdots \\ \dots\end{matrix}}{\mathrm{D}}\ {\scriptstyle \exists\mathrm{E}_\infty}$$

This infinitary rule is obviously inspired by the identification between the existential quantifier and an infinite disjunction, so the conflict between the disjunctive generality of $\dashv\mathrm{I}_\infty$ and the conjunctive generality of $\dashv\mathrm{E}_\infty$ is absent here.

Let us use this formulation to show that $\exists x A(x)$ is the strongest conclusion which follows from the I-rules for $\exists$:

$$\dfrac{\exists x A(x) \qquad \dfrac{[A(t_1)]^1}{\mathrm{C}}\,{\scriptstyle \bigstar_1} \qquad \begin{matrix}\dots\\ \vdots \\ \dots\end{matrix} \qquad \dfrac{[A(t_i)]^i}{\mathrm{C}}\,{\scriptstyle \bigstar_i} \qquad \begin{matrix}\dots\\ \vdots \\ \dots\end{matrix}}{\mathrm{C}}\ {\scriptstyle \exists\mathrm{E},\ 1,2,\dots}$$

---

[45] Indeed, this is what happens with the infinitary formulation of $\forall$ in note 48.

[46] This formulation of $\exists$E is considered in Read [2000]. We need to assume that every object has a name in our language, and that $t_1, \dots$ is a complete list of the terms of the language. I thank an anonymous referee for stressing this point. We will see that, using *criterion* 5.1 for 'full use' of rules, we can reject Steinberger's $\dashv$-rules in their finitary formulation and so circumvent this assumption, but that this approach leads to new troubles.

It is easy to see that we make full use both of $\exists E_\infty$ and of $\bigstar$.

Let us now show that $\exists x A(x)$ is the weakest major premise that can take part in $\exists E_\infty$. Given

$$\dfrac{C \qquad \begin{array}{c}[A(t_1)]\\ \vdots \\ D\end{array} \qquad \begin{array}{c}\dots\\ \vdots \\ \dots\end{array} \qquad \begin{array}{c}[A(t_i)]\\ \vdots \\ D\end{array} \qquad \begin{array}{c}\dots\\ \vdots \\ \dots\end{array}}{D}\,\blacktriangle$$

the proof of $C \vDash \exists x A(x)$ can be easily stated:

$$\dfrac{C \qquad \dfrac{[A(t_1)]^1}{\exists x A(x)}\,{\scriptstyle \exists I_1} \qquad \begin{array}{c}\dots\\ \vdots \\ \dots\end{array} \qquad \dfrac{[A(t_i)]^i}{\exists x A(x)}\,{\scriptstyle \exists I_i} \qquad \begin{array}{c}\dots\\ \vdots \\ \dots\end{array}}{\exists x A(x)}\,{\scriptstyle \blacktriangle,\ 1,2,\dots}$$

Also in this case, we make full use both of all I-rules for the existential quantifier,[47] and of the fact that C can be used as major premise in the E-rule for $\exists$ ($\blacktriangle$).[48]

While this infinitary treatment of the issue is surely very intuitive, it is not without problems. Indeed infinitary rules are very controversial, especially in an antirealistic framework. What would it mean for someone to grasp an infinite set of rules or, even worse, a single rule with an infinite number of premises? Moreover, since $\underline{\mathrm{h}}$*armony* can depend on the presentation of the rules, it may be possible for it to hold between $\exists I$ and $\exists E$ in

[47] It is important to notice that a single derivation of $\exists x A(x)$ from $A(t_i)$ for some t_i is not sufficient for the applicability of $\exists E$.

[48] For $\underline{\mathrm{h}}(\forall I_\infty, \forall E_\infty)$, we may impose $\forall E_\infty$ identical to $\exists E_\infty$, and adopt the I-rule

$$\dfrac{A(t_1) \qquad \cdots \qquad A(t_i) \qquad \cdots}{\forall x A(x)}\,{\scriptstyle \forall I}$$

so that the derivation

$$\begin{array}{c}
\dfrac{\dfrac{\forall x A(x) \text{ (or C)} \qquad \dfrac{[A(t_1)]^1 \qquad \cdots \qquad [A(t_1)]^i \qquad \cdots}{\text{C (or } \forall x A(x))}\,{\scriptstyle \bigstar \text{ (or } \forall I_\infty)}}{\text{C (or } \forall x A(x))}\,{\scriptstyle \forall E_1 \text{ (or } \blacktriangle_1),\ 1}}{} \\
\vdots \\
\dfrac{\forall x A(x) \text{ (or C)} \qquad \text{C (or } \forall x A(x))}{\text{C (or } \forall x A(x))}\,{\scriptstyle \forall E_i \text{ (or } \blacktriangle_i),\ i} \\
\vdots
\end{array}$$

fully uses both $\forall I_\infty$ and $\forall E_\infty$.

the standard finitary presentation of these rules, but not in their infinitary one.[49] Of course, the flaw that we displayed in the proof of $\underline{\text{h}}(\exists \text{I}_\infty, \exists \text{E}_\infty)$ cannot depend on their infinitary nature in the strict sense, since it would be an issue for finitary languages and models as well: just consider that our objection would still hold if there were just two terms in the language and two objects in the model. Nonetheless, Steinberger deals differently with the generality of ∃I and ∃E, and so we should discuss whether they are in <u>h</u>*armony* according to their original formulation.[50]

Let us try to address the issue of "full use" of clauses (a) and (b) for the quantificational rules in a finitary framework. While it is hard to find a complete adaptation of *criterion* 4.1 in this framework, the following seems to be at least an indisputable necessary condition for any rule to be fully applied:

Criterion 5.1 (Full use via rule-strength)**.** *If in the derivation that establishes (1) or (2) we could have used in place of Ⓢ I (Ⓢ E) a weaker rule Ⓢ I' (Ⓢ E'), then Ⓢ I (Ⓢ E) is not fully used in the derivation. If in the derivation that establishes (1) or (2) we could have used in place of the conditions described by Ⓢ I (Ⓢ E) the conditions described by a weaker rule Ⓢ I' (Ⓢ E'), then the conditions described by Ⓢ I (Ⓢ E) are not fully used in the derivation.*

Even though the notion of strength of a rule is not exempt from problems, it is far less controversial than that of full use.[51] Moreover, as we saw in section 1.1, this notion is already presupposed in the definition of *maximality*, so there is no reason not to apply it here.

It seems that this principle is sufficient to dismantle Steinberger's counterexample to Tennant's *harmony criterion*, even in its finitary formulation. Indeed, just consider the pair of rules

$$\frac{A(u)}{\exists x A(x)}\ {\scriptstyle \exists \mathrm{I}_u} \qquad\qquad \frac{\exists x A(x) \qquad \begin{matrix}[A(u)] \\ \vdots \\ \mathrm{D}\end{matrix}}{\mathrm{D}}\ {\scriptstyle \exists \mathrm{E}_u}$$

in which the term u does not occur schematically, that is which can be applied only for the term u. $\exists\mathrm{I}_u$ is part of the infinitary rule $\exists\mathrm{I}_\infty$ and $\exists\mathrm{E}_u$ is

[49]I thank an anonymous referee for suggesting this objection.

[50] Given its focus on strength, Tennant's version of *harmony seems* to be insensitive to differences in presentation (as opposed to other versions, like Read's one, of the same notion). Nonetheless, here even the legitimacy of the infinitary formulation is controversial, so we cannot relate to it to show that the finitary versions of ∃I and ∃E are not in *harmony* with each other. Moreover, we will soon show that presentation *is* relevant after all for Tennant's version of *harmony*, even though it is centered on strength.

[51]In section 1.3 we tried to settle some of the issues connected with rule-strength.

part of the infinitary rule $\Finv E_\infty$, but here they are considered as stand-alone rules. For every term there is just one finitary pair of finitary rules that deals with it: an introduction and an elimination rule. Of course, they are very weak and far weaker than the paradoxical rules proposed by Steinberger.[52] Indeed, they are obviously derivable from them and also clearly unable to make them admissible. Nonetheless, $\Finv E_u$ and the conditions described by $\Finv I_u$ are enough to prove that $\Finv xA(x)$ is the strongest conclusion possible under the conditions described by $\Finv$I. In the same way, $\Finv I_u$ and the conditions described by $\Finv E_u$ are enough strong to prove that $\Finv xA(x)$ is the weakest major premise possible under the conditions described by $\Finv$E.[53] Moreover, in every proof of clauses (1) and (2) of $\underline{\text{h}}$($\Finv$I,$\Finv$E) only one term u can occur in place of x in the discharged assumption, and the substitution of $\Finv$I and $\Finv$E with the respective pair of rules for this term results in a valid proof of the same conclusion. So we conclude that in Steinberger's proof of claims (1) and (2) $\Finv$-rules are not fully used, and that they are not in $\underline{\text{h}}$*armony* with each other. This blocks his objection.

Unfortunately, as we already mentioned, in the finitary framework the ∃-rules have the same flaws of the $\Finv$-rules: in the proofs of clauses (1) and (2) of $\underline{\text{h}}$(∃I,∃E) it is not possible to make full use of the rules, according to *criterion* 5.1. So this interpretation of clauses (a) and (b) is as useful to reply to Steinberger's objection, as damaging for Tennant's *criterion*.

Let us see the details of the problem with the finitary formulation of ∃I and ∃E. Consider the rule

$$\frac{A(a)}{\exists xA(x)}\ \scriptstyle\exists\text{I'}$$

formulated with the usual restrictions for a in ∀I.[54] In Steinberger's proof that $\exists xA(x)$ is the strongest conclusion possible under the conditions described by ∃I (1),[55] only the conditions described by ∃I' are really used. Indeed, since the restrictions imposed by ∃I' are needed for ∃E to be applicable, they are already observed. So, according to observation 5.1 and since ∃I' is far weaker than ∃I, there cannot be full use of the conditions described by ∃I in the proof of part (1) of $\underline{\text{h}}$(∃I,∃E). In the same way, in the proof that $\exists xA(x)$ is the weakest major premise possible under the conditions described by ∃E (2), ∃I' can be used in place of ∃I.[56] So the standard finitary

[52] Even though these rules come from a repackaging of $\Finv I_\infty$ and $\Finv E_\infty$, here the finitary constraint is observed, since there is no need to adopt all the rules belonging to the infinitary version.

[53] Be careful: while we use $\Finv I_u$ and $\Finv E_u$, we are still interested in the conditions described by $\Finv$I when we prove clause (1) and in the conditions described by $\Finv$E when we prove clause (2).

[54] That is, a shall not occur in any open assumption on which the premise depends.

[55] Which we saw in section 2.

[56] We could have adapted the counterexample used against the finitary version of the

rules for the existential quantifier are not in $\underline{h}$*armony* with each other, even though their infinitary formulations are, as we already established.[57]

6 From bad to worse

To sum up, Ⅎ-rules are $\underline{h}$*armonious* only in their finitary formulation and only according to *criterion* 4.1, so this is the only framework in which Steinberger's objection works. We can reject this counterexample both by giving an infinitary formulation of Ⅎ-rules, and by reinterpreting the full use clauses (a) and (b) according to *criterion* 5.1. Nonetheless, only in the infinitary framework the standard quantifiers $\exists$ and $\forall$ can be characterized by rules in $\underline{h}$*armony*, while the usual finitary version of them falls prey to the same objection we raised against Ⅎ-rules and so is unacceptable.

Hence, it seems that we have three alternatives:[58]

- To accept *criterion* 4.1 both for the propositional rules and for the infinitary reformulation of the quantificational rules, so obtaining a justification of intuitionistic logic;[59]
- To accept *criterion* 4.1 for the propositional rules and *criterion* 5.1 for the quantificational rules, on the basis that, even though in this way it is impossible to justify the full quantificational logic, it is still possible to reject Steinberger's Ⅎ-rules and so preserve the coherence of the system;
- To accept *criterion* 5.1 for both the propositional rules and the quantificational rules.

About this last alternative, we already saw that *criterion* 5.1 justifies only a very weak version of the quantificational rules. Let us evaluate its application in the field of propositional logic, by stepping back to the issue of quantum disjunction, which we displayed in section 1.2. In that section, we saw that $\underline{h}(\vee\text{I},\sqcup\text{E})$, $\underline{h}(\vee\text{I},\vee\text{E})$ and, since $\vee$E is stronger than $\sqcup$E, only $\vee$I and $\vee$E are in $\underline{H}$armony with each other.[60] Unfortunately, this cannot be correct if we extend *criterion* 5.1 of full use to the propositional fragment. Indeed,

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Ⅎ-rules, by selecting a pair of rules that suits the restrictions imposed by $\exists$E, but this approach seems more intuitive.

[57]As we already remarked in note 50, the relevance of the presentational aspects of the rules is far from obvious for Tennant's notion of *harmony*, and this makes the result even more interesting.

[58]I thank an anonymous referee for suggesting this development.

[59]While Tennant endorse relevant intuitionistic logic, relevance is not imposed via $\underline{H}$*armony*, but via other, extra restrictions; see Tennant [1997], section 10.9, especially p. 337.

[60]While there are some non obvious technicalities – evaluated in section 1.3 – needed to reach this conclusion even with *criterion* 4.1, this is surely at least Tennant's *desideratum*.

since to prove $\underline{\mathrm{h}}(\vee\mathrm{I},\sqcup\mathrm{E})$ and $\underline{\mathrm{h}}(\vee\mathrm{I},\vee\mathrm{E})$ we use the same derivations,[61] it is obvious that $\sqcup$E can be used to prove that $A \vee B$ is the strongest conclusion possible under the conditions described by $\vee$I (1) and that all the conditions described by $\sqcup$E are sufficient to prove that $A \vee B$ is the weakest major premise possible under the conditions described by $\vee$E (2). So, since $\vee$E is stronger than $\sqcup$E, it follows that neither $\vee$E, nor the conditions described by it are fully used in the proofs of clauses (1) and (2) of $\underline{\mathrm{h}}$*armony*, and after all $\vee$I is in $\underline{\mathrm{h}}$*armony* at most with $\sqcup$E, while this cannot be said for $\vee$I and $\vee$E. But the situation is even worse. Indeed, let us consider the following rules:[62]

$$\frac{\vdash A}{\vdash A \vee B}\,\vee\mathrm{I'} \qquad \frac{\vdash B}{\vdash A \vee B}\,\vee\mathrm{I'}$$

They are a weaker version of traditional I-rules for disjunction that is completely sufficient for our proof of clauses (1) and (2) of $\underline{\mathrm{h}}$*armony*. Hence, after all, it seems that even $\vee$I and $\sqcup$E are not in $\underline{\mathrm{h}}$*armony* with each other, since $\vee$I cannot be fully used in the proof of clauses (1) and (2). At most, what we can conclude is $\underline{\mathrm{h}}(\vee\mathrm{I'},\sqcup\mathrm{E})$, and even $\underline{\mathrm{H}}(\vee\mathrm{I'},\sqcup\mathrm{E})$, lacking stronger rules in $\underline{\mathrm{h}}$*armony* with either $\vee$I' or $\sqcup$E.

The lesson that we should learn from this example is that the request of *maximality* and the request of full use are frequently in contradiction to each other, if full use is interpreted as *criterion* 5.1 suggests. Indeed, if $\underline{\mathrm{h}}$(ⓈI,ⓈE), $\underline{\mathrm{h}}$(ⓈI',ⓈE), and ⓈI' is weaker than ⓈI but can nonetheless be used in the proofs of $\underline{\mathrm{h}}$(ⓈI,ⓈE), *maximality* would lead to accept $\underline{\mathrm{H}}$(ⓈI,ⓈE), while the request of full use leads to reject even $\underline{\mathrm{h}}$(ⓈI,ⓈE). Of course, the same holds for an elimination rule ⓈE' weaker than ⓈE. And since the request of full use is applied at the level of $\underline{\mathrm{h}}$*armony* – which is a prerequisite for *maximality* – it always wins the battle. Unfortunately, this leads to what seems an unacceptable weakening of the logical rules.[63]

Arguably, this conclusion leads to a rejection of the third alternative listed before, which consists in accepting *criterion* 5.1 for both the propositional and the quantificational rules. Someone could wonder nonetheless why this should be a problem for Tennant, since there are two other solutions to Steinberger's objection which keep, partially or completely, *criterion* 4.1 for full use.[64] I argue that there are three reasons why Tennant should worry about *criterion* 5.1: first of all, to expunge Steinberger's $\exists$-rules using *criterion* 4.1 we need an infinitary reformulation of the quantificational rules, which is at odds with the antirealist theories of meaning; as a second point,

[61] See note 8.

[62] Which we formulate using sequents in order to improve their readability.

[63] As we observed in section 1.4, Tennant tries to divorce full use from *maximality*, in contrast to standard proof-theoretic semantics, which applies both notions inside stability. When the relation between these notions is rectified, chickens are coming home to roost.

[64] I thank an anonymous referee for stressing this issue.

we lack a non-*ad-hoc* rejection of *criterion* 5.1 as a *criterion* of full use, which does not rest on its performance to select h*armonious* rules;[65] as a last point, lacking a good argument in this direction, accepting *criterion* 4.1 for propositional rules and *criterion* 5.1 for the quantificational rules is just cherry-picking.

In order to develop further the last two points, let us confront the alternative *criteria* of full use, by focusing first of all on the propositional case, and especially on disjunction. The difference between *criteria* 4.1 and 5.1 is that the first one evaluates full use *without* taking in consideration side assumptions. In other words, *criterion* 4.1 is intentionally designed to be a low resolution *criterion* of full use. On the contrary, *criterion* 5.1 is much more fine-grained, since it implicitly considers any aspect of the rules under investigation, by encouraging comparison with *any* alternative rule applicable in the proof of claims (1) and (2).[66] The same conclusion can be reached by focusing on the quantificational rules. As an example, let us consider the evaluation of $\exists$-rules. The main difference between applying *criterion* 4.1 or *criterion* 5.1 is that the first ignores completely any conditions on the substitution of terms, while the second is sensitive to this difference. Of course, fine-grained principles are always preferable, at least *prima facie*, and so, even though Tennant could defend his *criterion* in some way,[67] the burden of proof is surely on him. Moreover, if he wants to use *criterion* 5.1 for the quantificational rules, in order to solve Steinberger's objection in a finitary framework, and then step back to *criterion* 4.1 for the propositional fragment, he has to explain why terms substitution should be taken into consideration for full use, but side assumptions should not.

It could be argued that superior granularity is sometimes preferable in a *criterion.* As an example, the identity *criterion* for proofs based on β-η reductions, proposed by Prawitz following a suggestion of Martin-Löf,[68] has been questioned by Widebäck for being too much fine-grained, distinguishing between "strikingly similar proofs".[69] The author even concludes that "it is also possible that [...] the studied notion of identity *in being too delicate* is a rather uninteresting notion with respect to our informal motivation."[70] So unwanted consequences may lead to rejecting a certain *criterion* because it is regarded as excessively fine-grained, depending on which ideas or concepts one intends to characterize.[71]

[65]As we will see, we have on the contrary good evidence that this is to be preferred to Tennant's version.

[66]Indeed it is no coincidence that such a notion of full use is much nearer to what is usually applied in the definition of stability; see section 1.4.

[67]As an example, Tennant could decide that, after all, what he wants is not full use.

[68]Prawitz [1971], p. 257.

[69]Widebäck [2001], p. 17.

[70]Widebäck [2001], p. 17, *italics* mine.

[71]I really thank an anonymous referee for suggesting this defence and providing these examples.

The argument is sound, and surely could constitute a treat for *criterion* 5.1, so let us try to apply it to our issue. The notion that we want to characterize is that of full use, and we want it to provide a justification of a plausible logical system that is neither too weak nor so strong to be trivial. So, there are two *desiderata* for our *criteria*: to appropriately characterize the intuitive notion of full use (like the identity *criterion* for proofs based on β-η reductions should characterize the intuitive notion of identity of proofs); to serve as a tool to justify a plausible logical system. Let us evaluate *criterion* 4.1 and *criterion* 5.1 according to this two *desiderata*.

Accordingly, there is not a shared strong intuitive notion of full use for the rules of logic. Hence the first point is difficult to judge. Nonetheless, it seems to me that our previous declaration that a more fine-grained *criterion* should be considered *prima facie* preferable is still sound, even though severely weakened. Another evidence in favor of *criterion* 5.1 is that Tennant's *criterion* 4.1 is the only one to sever full use from maximality and rule-strength, at least in proof-theoretic semantics. Indeed, the standard treatment of full use *is* via rule strength. So, it seems that at least a big part of the community working on this issue shares with me the intuition that these notions should not be fully divorced as Tennant suggests.[72]

Admittedly, the previous argument is blatantly insufficient to select one of the proposed *criteria*, so let us check the second *desideratum*: the election of an appropriate logical system. As we argued extensively, infinitary frameworks are highly controversial in antirealist theories of meaning, so we will confine ourselves to finitary ones. This means that *criterion* 4.1 leads to triviality, via Steinberger's rules for $\sqsupset$. On the contrary *criterion* 5.1 leads to a serious weakening of the logical system. None of these alternatives is well accepted, but the first is the trademark of logical failure. So, if I had to choose between these two *criteria*, with a gun on my head, I would opt for the second one. Moreover, as we already stressed, *criterion* 5.1 leads to the justification of fully respectable logical systems, at least inside other more traditional approaches to proof-theoretic semantics. What seems to cause trouble in Tennant's definition of harmony, when formulated with *criterion* 5.1, is not that full use is used to define maximality, but that it already occurs in the definition of $\underline{\text{h}}$armony. Nothing of this kind can be said about *criterion* 4.1, which has no application outside Tennant's works. So, even considering our second *desideratum*, *criterion* 5.1 seems to behave better than *criterion* 4.1 both inside Tennant's framework and in general.

As previously anticipated, there is a third alternative that suits better the *desideratum* of selecting an appropriate logic, but whose intuitiveness is difficult to evaluate: adopting *criterion* 4.1 for propositional rules and *criterion* 5.1 for the quantificational rules. Indeed, even though this combination of *criteria* still cause some unwanted weakening in the quantifi-

[72]See section 1.4 for more details.

cational rules, it at least justifies the full intuitionistic propositional logic and does not lead to trivialism. The main problem with this alternative is that the shift from *criterion* 4.1 to *criterion* 5.1 in the transition from propositional to quantificational logic seems *ad hoc*. Maybe, the selection of a decent, even though not optimal, logic could be enough to conclude that terms substitution should be taken into consideration for full use, but side assumptions should not. However, the intuitiveness of the formal *criterion* of full use still is a *desideratum*, so we have to investigate this issue more thoroughly for the combination of *criterion* 4.1 and *criterion* 5.1. And that such a composite *criterion* could be intuitively and theoretically palatable is far from obvious.

In conclusion, the only viable alternatives are quite miserable: adopting *criterion* 5.1 both for propositional and quantificational rules, so obtaining a very weak but at least coherent system; or adopting *criterion* 4.1 for the propositional rules and *criterion* 5.1 for quantificational rules, so obtaining a more interesting logical system, but owing a clarification of this combination of *criteria*. It is important to remark that both solutions block Steinberger's objection. Moreover, any of these alternatives poses a serious threat to Tennant's definition of harmony, so the main points of this paper are independent of the preferred notion of full use.[73] Hence, the choice is up to Tennant. If he wants to justify at least full propositional intuitionistic logic, he has to provide a detailed account of how he interprets full use, and what reasons he has to do so. Otherwise, he can adopt *criterion* 5.1 if he is happy with a powerless logical system. Both alternatives, it seems to me, are highly unappealing.

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[73] I thank an anonymous referee for stressing this point.

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