Peano's Counterexample to Harmony

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July 18, 2019

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Natural deduction as a **theory of meaning**.

- **Introduction** rules are **meaning conferring**;
- **Elimination** rules are justified by introduction rules.

"The introductions represent, as it were, the 'definitions' of the symbols concerned, and the eliminations are no more, in the final analysis, than the consequences of these definitions. This fact may be expressed as follows: In eliminating a symbol, we may use the formula with whose terminal symbol we are dealing only 'in the sense afforded it by the introduction of that symbol'." (Gentzen, Investigation into Logical Deduction, 1934/35, 5.13)

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Harmony: I-rules and E-rules are in harmony iff , when we have the **major premise of an E-rule derived using an I-rule**, then we can **reduce the proof**.

An E-rule is justified iff it is in harmony with an I-rule.

"Let α be an application of an elimination rule that has B as consequence. Then, deductions that satisfy the sufficient condition $[\cdots]$ for deriving the major premiss of α, when combined with deductions of the minor premisses of α (if any), already "contain" a deduction of B; the deduction of B is thus obtainable directly from the given deductions without the addition of α." (Prawitz, Natural Deduction, 1965, p. 33)

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$$\begin{array}{c}\dfrac{\dfrac{\begin{array}{c}[A]\\ \vdots\,\Phi_1\\ B\end{array}}{A \to B}\,{\to}I \qquad \begin{array}{c}\vdots\,\Phi_2\\ A\end{array}}{B}\,{\to}E\\ \vdots\,\Phi_3\end{array} \quad \rightsquigarrow \quad \begin{array}{c}\vdots\,\Phi_2\\ [A]\\ \vdots\,\Phi_1\\ B\\ \vdots\,\Phi_3\end{array}$$

Harmony of intuitionistic $\to$

$$
\dfrac{\dfrac{\begin{array}{c}[A]\\ \vdots\,\Phi_1\\ B\end{array}}{A \to B}\to I \qquad \begin{array}{c}\vdots\,\Phi_2\\ A\end{array}}{\begin{array}{c}B\\ \vdots\,\Phi_3\end{array}}\to E
\quad \rightsquigarrow \quad
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$$

Harmony of intuitionistic $\to$

$$
\dfrac{\dfrac{\begin{matrix}[A] \\ \vdots\,\Phi_1 \\ B\end{matrix}}{A \to B}\to I \qquad \begin{matrix}\vdots\,\Phi_2 \\ A\end{matrix}}{\begin{matrix}B \\ \vdots\,\Phi_3\end{matrix}}\to E
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\quad \rightsquigarrow \quad
\begin{array}{c}\vdots\,\Phi_2\\ [A]\\ \vdots\,\Phi_1\\ B\\ \vdots\,\Phi_3\end{array}
$$

Prior's tonk

$$Itonk\ \frac{A}{AtonkB} \qquad Etonk\ \frac{AtonkB}{B}$$

$$\begin{array}{r}Itonk\\Etonk\end{array}\ \dfrac{\dfrac{A}{AtonkB}}{B} \quad \rightsquigarrow \quad ?$$

- *Tonk* is not an harmonious connective, and indeed it leads to triviality in standard logical systems.

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Conservative extension

- $\mathfrak{S}$ formulated in the language $\mathfrak{L}$;
- $\mathfrak{S}'$ formulated in the language $\mathfrak{L}'$;
- $\mathfrak{L} \subset \mathfrak{L}'$;
- $\mathfrak{S} \subset \mathfrak{S}'$.

$\mathfrak{S}'$ **conservatively extends** $\mathfrak{S}$ iff$_{def}$ $\Gamma \vdash_{\mathfrak{S}'} C$ and $\Gamma, C \in \mathfrak{L}$ entail $\Gamma \vdash_{\mathfrak{S}} C$.

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I

$$I\wedge\ \frac{A \qquad B}{A \wedge B} \qquad E\wedge\ \frac{A \wedge B}{A} \qquad E\wedge\ \frac{A \wedge B}{A} \qquad I\vee\ \frac{A}{A \vee B} \qquad I\vee\ \frac{B}{A \vee B}$$

$$E\vee\ \frac{A \vee B \qquad \begin{array}{c}[A]\\ \vdots \\ C\end{array} \qquad \begin{array}{c}[B]\\ \vdots \\ C\end{array}}{C}$$

$$I\supset\ \frac{\begin{array}{c}[A]\\ \vdots \\ B\end{array}}{A \supset B} \qquad E\supset\ \frac{A \supset B \qquad A}{B} \qquad I\neg\ \frac{\begin{array}{c}[A]\\ \vdots \\ \bot\end{array}}{\neg A} \qquad E\neg\ \frac{\neg A \qquad A}{\bot} \qquad Efq\ \frac{\bot}{C}$$

Separability of I

- **I** formulated in the language $\wedge \vee \rightarrow \neg \bot$;
- $\{\oplus, \ldots, \otimes\} \subset \{\wedge \vee \rightarrow \neg \bot\}$;
- $\mathbf{I}^{\oplus, \ldots, \otimes}$ is the fragment of **I** with only **the rules for** $\oplus, \ldots, \otimes$.

Conservativeness for I: I conservatively extends $\mathbf{I}^{\oplus, \ldots, \otimes}$.

Separability for I: an intuitionistic theorem can be proved in **I** using only the rules for the connectives that occur in it.

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$$\neg\neg E\ \frac{\neg\neg A}{A}$$

Non separability of C

Since:

- $\mathbf{C}^{\wedge\vee\rightarrow\perp}$ is identical to $\mathbf{I}^{\wedge\vee\rightarrow\perp}$;
- $\vdash_{\mathbf{C}} ((A \rightarrow B) \rightarrow A) \rightarrow A$ but $\nvdash_{\mathbf{I}} ((A \rightarrow B) \rightarrow A) \rightarrow A$

Non-Conservativeness for C: C non-conservatively extends $\mathbf{C}^{\wedge\vee\rightarrow\perp}$.

Non-Separability for C: $((A \rightarrow B) \rightarrow A) \rightarrow A$ can be proved in **C only using** the classical rules for **negation**.

Since:

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Harmony and Conservativeness

- $\mathfrak{S}$ formulated in the language $\mathfrak{L}$;
- $\mathfrak{S}'$ formulated in the language $\mathfrak{L}'$;
- $\mathfrak{L} \subset \mathfrak{L}'$;
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Conjecture 1: Both $\mathfrak{S}$ and $\mathfrak{S}'$ **are harmonious** sets of rules $\Rightarrow$ $\mathfrak{S}'$ **conservatively extends** $\mathfrak{S}$.

Conjecture 2: $\mathfrak{S}'$ **conservatively extends** $\mathfrak{S}$ $\Rightarrow$ both $\mathfrak{S}$ and $\mathfrak{S}'$ can be characterised by **harmonious** sets of rules .

Conservativeness $\not\Rightarrow$ Harmony: $\mathbf{I}^{\wedge\vee\neg\bot tonk}$ is conservatively extended by the rules for implication, but neither of these systems is harmonious.

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First Counterexample and Stability

Quantum disjunction

$\mathbf{Q} =_{def}$ **I** with $\vee$-rules substituted by $\sqcup$-rules.

$$
I\sqcup\ \frac{\Gamma \vdash A}{\Gamma \vdash A \sqcup B} \qquad I\sqcup\ \frac{\Gamma \vdash B}{\Gamma \vdash A \sqcup B} \qquad E\sqcup\ \frac{\Gamma \vdash A \sqcup B \qquad A \vdash C \qquad B \vdash C}{\Gamma \vdash C}
$$

$\sqcup$-rules, $\wedge$-rules and intuitionistic $\rightarrow$-rules are harmonious.

But $A \wedge (B \sqcup C) \nvdash_{\mathbf{Q}^{\wedge\sqcup}} (A \wedge B) \sqcup (A \wedge C)$ and
$A \wedge (B \sqcup C) \vdash_{\mathbf{Q}^{\wedge\sqcup\rightarrow}} (A \wedge B) \sqcup (A \wedge C)$.

$$
\begin{prooftree}
\AxiomC{$A \wedge (B \sqcup C)$}
\UnaryInfC{$B \sqcup C$}
\AxiomC{$[A \wedge (B \sqcup C)]^1$}
\UnaryInfC{A}
\AxiomC{$[B]^2$}
\BinaryInfC{$A \wedge B$}
\UnaryInfC{$(A \wedge B) \sqcup (A \wedge C)$}
\RightLabel{$\rightarrow I_1$}
\UnaryInfC{$(A \wedge (B \sqcup C)) \rightarrow ((A \wedge B) \sqcup (A \wedge C))$}
\AxiomC{$[A \wedge (B \sqcup C)]^1$}
\UnaryInfC{A}
\AxiomC{$[C]^2$}
\BinaryInfC{$A \wedge C$}
\UnaryInfC{$(A \wedge B) \sqcup (A \wedge C)$}
\RightLabel{$\rightarrow I_1$}
\UnaryInfC{$(A \wedge (B \sqcup C)) \rightarrow ((A \wedge B) \sqcup (A \wedge C))$}
\RightLabel{$\sqcup E_2$}
\TrinaryInfC{$(A \wedge (B \sqcup C)) \rightarrow ((A \wedge B) \sqcup (A \wedge C))$}
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\BinaryInfC{$(A \wedge B) \sqcup (A \wedge C)$}
\end{prooftree}
$$

Q $=_{def}$ **I** with $\vee$-rules substituted by $\sqcup$-rules.

$$I\sqcup\ \frac{\Gamma \vdash A}{\Gamma \vdash A \sqcup B} \qquad I\sqcup\ \frac{\Gamma \vdash B}{\Gamma \vdash A \sqcup B} \qquad E\sqcup\ \frac{\Gamma \vdash A \sqcup B \qquad A \vdash C \qquad B \vdash C}{\Gamma \vdash C}$$

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$$\cfrac{\cfrac{\cfrac{A \wedge (B \sqcup C)}{B \sqcup C} \qquad \cfrac{\cfrac{\cfrac{\cfrac{[A \wedge (B \sqcup C)]^1}{A} \qquad [B]^2}{A \wedge B}}{(A \wedge B) \sqcup (A \wedge C)}}{(A \wedge (B \sqcup C)) \rightarrow ((A \wedge B) \sqcup (A \wedge C))}{\scriptstyle \rightarrow I_1} \qquad \cfrac{\cfrac{\cfrac{\cfrac{[A \wedge (B \sqcup C)]^1}{A} \qquad [C]^2}{A \wedge C}}{(A \wedge B) \sqcup (A \wedge C)}}{(A \wedge (B \sqcup C)) \rightarrow ((A \wedge B) \sqcup (A \wedge C))}{\scriptstyle \rightarrow I_1}}{(A \wedge (B \sqcup C)) \rightarrow ((A \wedge B) \sqcup (A \wedge C))}{\scriptstyle \sqcup E_2} \qquad A \wedge (B \sqcup C)}{(A \wedge B) \sqcup (A \wedge C)}$$

$\mathbf{Q} =_{def}$ **I** with $\vee$-rules substituted by $\sqcup$-rules.

$$I\sqcup\ \frac{\Gamma \vdash A}{\Gamma \vdash A \sqcup B} \qquad I\sqcup\ \frac{\Gamma \vdash B}{\Gamma \vdash A \sqcup B} \qquad E\sqcup\ \frac{\Gamma \vdash A \sqcup B \qquad A \vdash C \qquad B \vdash C}{\Gamma \vdash C}$$

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$\sqcup$-rules, $\wedge$-rules and intuitionistic $\rightarrow$-rules are harmonious.

But $A \wedge (B \sqcup C) \nvdash_{\mathbf{Q}^{\wedge\sqcup}} (A \wedge B) \sqcup (A \wedge C)$ and
$A \wedge (B \sqcup C) \vdash_{\mathbf{Q}^{\wedge\sqcup\rightarrow}} (A \wedge B) \sqcup (A \wedge C)$.

$$\dfrac{\dfrac{\dfrac{A \wedge (B \sqcup C)}{B \sqcup C} \qquad \dfrac{\dfrac{\dfrac{\dfrac{[A \wedge (B \sqcup C)]^1}{A} \qquad [B]^2}{A \wedge B}}{(A \wedge B) \sqcup (A \wedge C)}}{(A \wedge (B \sqcup C)) \rightarrow ((A \wedge B) \sqcup (A \wedge C))}{\scriptstyle \rightarrow I_1} \qquad \dfrac{\dfrac{\dfrac{\dfrac{[A \wedge (B \sqcup C)]^1}{A} \qquad [C]^2}{A \wedge C}}{(A \wedge B) \sqcup (A \wedge C)}}{(A \wedge (B \sqcup C)) \rightarrow ((A \wedge B) \sqcup (A \wedge C))}{\scriptstyle \rightarrow I_1}}{(A \wedge (B \sqcup C)) \rightarrow ((A \wedge B) \sqcup (A \wedge C))}{\scriptstyle \sqcup E_2} \qquad A \wedge (B \sqcup C)}{(A \wedge B) \sqcup (A \wedge C)}$$

Quantum disjunction

$\mathbf{Q} =_{def}$ **I** with $\vee$-rules substituted by $\sqcup$-rules.

$$I\sqcup \; \frac{\Gamma \vdash A}{\Gamma \vdash A \sqcup B} \qquad I\sqcup \; \frac{\Gamma \vdash B}{\Gamma \vdash A \sqcup B} \qquad E\sqcup \; \frac{\Gamma \vdash A \sqcup B \qquad A \vdash C \qquad B \vdash C}{\Gamma \vdash C}$$

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Stability: $\oplus$I and $\oplus$E are stable iff, they are in harmony and $\oplus$E completely uses the meaning given to $\oplus$ by $\oplus$I.

Inverse harmony: "Whatever follows from the direct grounds for deriving a proposition must follow from that proposition." (Negri & von Plato, *Structural Proof Theory*, 2001, p. 6.)

Inverse harmony (Lorenzen): if $p_1 \Rightarrow p_0; \ldots; p_n \Rightarrow p_0$, then $[p_1 \Rightarrow p; \ldots; p_n \Rightarrow p] \Rightarrow (p_0 \Rightarrow p)$.

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Jacinto & Read: If C can be derived from the **direct grounds** for $A \oplus B$ together with the assumptions Γ_i with $1 \leqslant i \leqslant m$, then C can be derived from $A \oplus B$ together with the assumptions $\bigcup_{i=1}^{m} \Gamma_i$ by appealing only to the first derivations (one for each ground) and $\oplus$E.

$$
\begin{array}{ccc}
\Gamma_1 \quad A & & \Gamma_2 \quad B \\
\vdots \Phi_1 & & \vdots \Phi_2 \\
C & & C
\end{array}
$$

$$\Downarrow$$

$$
\dfrac{A \vee B \qquad \begin{array}{c}\Gamma_1 \quad [A]^1 \\ \vdots \Phi_1 \\ C\end{array} \qquad \begin{array}{c}\Gamma_2 \quad [B]^1 \\ \vdots \Phi_2 \\ C\end{array}}{C}\ \vee\mathsf{E}_1
$$

Formal Requirements for Stability

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$$\begin{array}{c} \Gamma_1 \quad A \\ \vdots \Phi_1 \\ C \end{array} \qquad\qquad \begin{array}{c} \Gamma_2 \quad B \\ \vdots \Phi_2 \\ C \end{array}$$

$$\Downarrow$$

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$$
\begin{array}{c}
\Gamma_1 \qquad A \\
\vdots\Phi_1 \\
C
\end{array}
\qquad\qquad\qquad
\begin{array}{c}
\Gamma_2 \qquad B \\
\vdots\Phi_2 \\
C
\end{array}
$$

$$\Downarrow$$

$$
\vee\mathsf{E}_1 \; \dfrac{A \vee B \qquad
\begin{array}{c}
\Gamma_1 \qquad [A]^1 \\
\vdots\Phi_1 \\
C
\end{array}
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- $\mathfrak{S}$ formulated in the language $\mathfrak{L}$;
- $\mathfrak{S}'$ formulated in the language $\mathfrak{L}'$;
- $\mathfrak{L} \subset \mathfrak{L}'$;
- $\mathfrak{S} \subset \mathfrak{S}'$;

Conjecture 1': if $\mathfrak{S}$ is formulated using **stable rules**, and $\mathfrak{S}'$ is obtained adding to $\mathfrak{S}$ only **harmonious rules**, then $\mathfrak{S}'$ is a conservative extension of $\mathfrak{S}$.(Dummett, *The Logical Basis of Metaphysics*, 1991, p. 290)

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Truth Predicate and its Problems

Truth predicate

$$\mathscr{T}\mathsf{I}\ \frac{A}{\mathscr{T}(\ulcorner A \urcorner)} \qquad \mathscr{T}\mathsf{E}\ \frac{\mathscr{T}(\ulcorner A \urcorner)}{A}$$

They are harmonious and stable, but lead to a non-conservative extension of **PA**.

$$\vdash_{\mathbf{PA}+\mathscr{T}} \mathscr{G}_{PA}$$

but

$$\nvdash_{\mathbf{PA}} \mathscr{G}_{PA}.$$

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Problems with this counterexample

Conjecture 1': The extension of a stable system with harmonious rules is conservative.

$\mathscr{T}$-rules are harmonious and stable, but what about **PA**?

The extension is conservative if we do not allow $\mathscr{T}$ to occur in **induction schema**.

Extra: What lesson should we learn? We want harmony or conservativeness?

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Peano's Counterexample

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The definition $(a/b)?(c/d) =_{def} (a+c)/(b+d)$ is unacceptable!

$$?\mathrm{I}\ \frac{(a+c)/(b+d) = e/f}{(a/b)?(c/d) = e/f} \qquad ?\mathrm{E}\ \frac{(a/b)?(c/d) = e/f}{(a+c)/(b+d) = e/f}$$

$$\text{Add.}\ \dfrac{?\mathrm{E}\ \dfrac{\text{Sub. of Id.}\ \dfrac{?\mathrm{I}\ \dfrac{(1+1)/(2+3) = 2/5}{(1/2)?(1/3) = 2/5} \qquad 1/2 = 2/4}{(2/4)?(1/3) = 2/5}}{(2+1)/(4+3) = 2/5}}{3/7 = 2/5}$$

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Harmony and stability

$$\dfrac{\dfrac{\begin{array}{c}\vdots\\(a+c)/(b+d)=e/f\end{array}}{(a/b)?(c/d)=e/f}\,?\mathrm{I}}{\begin{array}{c}(a+c)/(b+d)=e/f\\\vdots\end{array}}\,?\mathrm{E} \quad\rightsquigarrow\quad \begin{array}{c}\vdots\\(a+c)/(b+d)=e/f\\\vdots\end{array}$$

Stability: E? is in harmony with I? and it is obviously the strongest such rule.

$$
\dfrac{\dfrac{\begin{matrix}\vdots \\ (a+c)/(b+d) = e/f\end{matrix}}{(a/b)?(c/d) = e/f}\,{?\mathrm{I}}}{\begin{matrix}(a+c)/(b+d) = e/f \\ \vdots\end{matrix}}\,{?\mathrm{E}}
\quad \rightsquigarrow \quad
\begin{matrix}\vdots \\ (a+c)/(b+d) = e/f \\ \vdots\end{matrix}
$$

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$$
\cfrac{\cfrac{\vdots}{(a+c)/(b+d) = e/f}}{\dfrac{(a/b)?(c/d) = e/f}{\dfrac{(a+c)/(b+d) = e/f}{\vdots}}\ {\scriptstyle ?E}}\ {\scriptstyle ?I}
\quad \rightsquigarrow \quad
\begin{array}{c} \vdots \\ (a+c)/(b+d) = e/f \\ \vdots \end{array}
$$

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$$\dfrac{(a/b)?(c/d) = e/f \qquad \begin{matrix}[(a+c)/(b+d) = e/f] \\ \vdots \\ C\end{matrix}}{C}\ ?\mathrm{E}$$

$$\dfrac{\dfrac{\begin{matrix}\vdots \\ (a+c)/(b+d) = e/f\end{matrix}}{(a/b)?(c/d) = e/f}\ ?\mathrm{I} \qquad \begin{matrix}[(a+c)/(b+d) = e/f] \\ \vdots \\ C\end{matrix}}{C}\ ?\mathrm{E} \quad \rightsquigarrow \quad \begin{matrix}\vdots \\ (a+c)/(b+d) = e/f \\ \vdots \\ C\end{matrix}$$

$$
?\mathrm{E}\ \dfrac{(a/b)?(c/d) = e/f \qquad \begin{array}{c}[(a+c)/(b+d) = e/f] \\ \vdots \\ C\end{array}}{C}
$$

$$
?\mathrm{E}\ \dfrac{?\mathrm{I}\ \dfrac{\begin{array}{c}\vdots \\ (a+c)/(b+d) = e/f\end{array}}{(a/b)?(c/d) = e/f} \qquad \begin{array}{c}[(a+c)/(b+d) = e/f] \\ \vdots \\ C\end{array}}{C}
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$$

$$
\dfrac{(a/b)?(c/d) = e/f \qquad \begin{matrix} [(a+c)/(b+d) = e/f] \\ \vdots \\ C \end{matrix}}{C}\ ?\mathsf{E}
$$

$$
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$$

And old objections?

Starting system: can we non-conservatively extend a **stable system** with '?'?

Which lesson: can we conclude something about **acceptability** or not of **one of the two principles**?

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Minimal logic

$$\mathfrak{s}\mathsf{I}\ \frac{m = n}{\mathfrak{s}(m) = \mathfrak{s}(n)} \qquad \mathfrak{s}\mathsf{E}\ \frac{\mathfrak{s}(m) = \mathfrak{s}(n)}{m = n} \qquad \bot\mathsf{I}\ \frac{\mathfrak{s}(n) = 0}{\bot} \qquad \bot\mathsf{E}\ \frac{\bot}{\mathfrak{s}(n) = 0}$$

$$+0\mathsf{I}\ \frac{m = n}{m + 0 = n} \qquad +0\mathsf{E}\ \frac{m + 0 = n}{m = n} \qquad +\mathsf{I}\ \frac{\mathfrak{s}(m + n) = l}{m + \mathfrak{s}(n) = l} \qquad +\mathsf{E}\ \frac{m + \mathfrak{s}(n) = l}{\mathfrak{s}(m + n) = l}$$

$$\times 0\mathsf{I}\ \frac{0 = n}{m \times 0 = n} \qquad \times 0\mathsf{E}\ \frac{m \times 0 = n}{0 = n} \qquad \times\mathsf{I}\ \frac{(m \times n) + m = l}{m \times \mathfrak{s}(n) = l} \qquad \times\mathsf{E}\ \frac{m \times \mathfrak{s}(n) = l}{(m \times n) + m = l}$$

$$\begin{array}{c} [F(a)] \\ \vdots \\ =\mathsf{I}\ \dfrac{F(b)}{a = b} \end{array} \qquad =\mathsf{E}\ \frac{a = b \qquad A(a)}{A(b)}$$

Restrictions on $=$I: F has to be a fully general *predicative variable* which does not occur in other open assumptions.

Note: $=$E enables the substitution of any number of occurrences of a in $A(a)$.

Minimal logic

$$\mathfrak{s}\mathsf{I}\ \frac{m = n}{\mathfrak{s}(m) = \mathfrak{s}(n)} \qquad \mathfrak{s}\mathsf{E}\ \frac{\mathfrak{s}(m) = \mathfrak{s}(n)}{m = n} \qquad \bot\mathsf{I}\ \frac{\mathfrak{s}(n) = 0}{\bot} \qquad \bot\mathsf{E}\ \frac{\bot}{\mathfrak{s}(n) = 0}$$

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$$\mathfrak{s}\mathsf{I}\ \frac{m = n}{\mathfrak{s}(m) = \mathfrak{s}(n)} \qquad \mathfrak{s}\mathsf{E}\ \frac{\mathfrak{s}(m) = \mathfrak{s}(n)}{m = n} \qquad \bot\mathsf{I}\ \frac{\mathfrak{s}(n) = 0}{\bot} \qquad \bot\mathsf{E}\ \frac{\bot}{\mathfrak{s}(n) = 0}$$

$$+0\mathsf{I}\ \frac{m = n}{m + 0 = n} \qquad +0\mathsf{E}\ \frac{m + 0 = n}{m = n} \qquad +\mathsf{I}\ \frac{\mathfrak{s}(m + n) = l}{m + \mathfrak{s}(n) = l} \qquad +\mathsf{E}\ \frac{m + \mathfrak{s}(n) = l}{\mathfrak{s}(m + n) = l}$$

$$\times 0\mathsf{I}\ \frac{0 = n}{m \times 0 = n} \qquad \times 0\mathsf{E}\ \frac{m \times 0 = n}{0 = n} \qquad \times\mathsf{I}\ \frac{(m \times n) + m = l}{m \times \mathfrak{s}(n) = l} \qquad \times\mathsf{E}\ \frac{m \times \mathfrak{s}(n) = l}{(m \times n) + m = l}$$

$$=\mathsf{I}\ \frac{\begin{matrix}[F(a)] \\ \vdots \\ F(b)\end{matrix}}{a = b} \qquad =\mathsf{E}\ \frac{a = b \qquad A(a)}{A(b)}$$

Restrictions on $=$I: F has to be a fully general *predicative variable* which does not occur in other open assumptions.

Note: $=$E enables the substitution of any number of occurrences of *a* in *A(a)*.

Minimal logic

$$\mathfrak{s}\mathsf{I}\ \frac{m = n}{\mathfrak{s}(m) = \mathfrak{s}(n)} \qquad \mathfrak{s}\mathsf{E}\ \frac{\mathfrak{s}(m) = \mathfrak{s}(n)}{m = n} \qquad \bot\mathsf{I}\ \frac{\mathfrak{s}(n) = 0}{\bot} \qquad \bot\mathsf{E}\ \frac{\bot}{\mathfrak{s}(n) = 0}$$

$$+0\mathsf{I}\ \frac{m = n}{m + 0 = n} \qquad +0\mathsf{E}\ \frac{m + 0 = n}{m = n} \qquad +\mathsf{I}\ \frac{\mathfrak{s}(m + n) = l}{m + \mathfrak{s}(n) = l} \qquad +\mathsf{E}\ \frac{m + \mathfrak{s}(n) = l}{\mathfrak{s}(m + n) = l}$$

$$\times 0\mathsf{I}\ \frac{0 = n}{m \times 0 = n} \qquad \times 0\mathsf{E}\ \frac{m \times 0 = n}{0 = n} \qquad \times\mathsf{I}\ \frac{(m \times n) + m = l}{m \times \mathfrak{s}(n) = l} \qquad \times\mathsf{E}\ \frac{m \times \mathfrak{s}(n) = l}{(m \times n) + m = l}$$

$$=\mathsf{I}\ \frac{\begin{array}{c}[F(a)] \\ \vdots \\ F(b)\end{array}}{a = b} \qquad =\mathsf{E}\ \frac{a = b \qquad A(a)}{A(b)}$$

Restrictions on $=$I: F has to be a fully general *predicative variable* which does not occur in other open assumptions.

Note: $=$E enables the substitution of any number of occurrences of a in $A(a)$.

Harmony of =-rules

$$
\dfrac{\dfrac{\dfrac{\begin{matrix}[F(a)] \\ \vdots\,\Phi_1\end{matrix}}{F(b)}}{a = b}\,{=}\mathrm{I} \qquad A(a)}{A(b)}\,{=}\mathrm{E}
\quad \rightsquigarrow \quad
\begin{matrix} A(a) \\ \vdots\,\Phi_1^* \\ A(b) \end{matrix}
$$

Harmony of =-rules

$$
\dfrac{\dfrac{\begin{array}{c}[F(a)] \\ \vdots \Phi_1 \\ F(b)\end{array}}{a = b}\;{=}\mathsf{I} \qquad A(a)}{A(b)}\;{=}\mathsf{E}
\qquad \rightsquigarrow \qquad
\begin{array}{c} A(a) \\ \vdots \Phi_1^* \\ A(b) \end{array}
$$

$$
\dfrac{\dfrac{\begin{array}{c}[F(a)] \\ \vdots\Phi_1 \\ F(b)\end{array}}{a = b}{=}\mathsf{I} \qquad A(a)}{A(b)}{=}\mathsf{E}
\quad \rightsquigarrow \quad
\begin{array}{c} A(a) \\ \vdots\Phi_1^* \\ A(b) \end{array}
$$

Harmony of =-rules

$$
\dfrac{\dfrac{\begin{array}{c}[F(a)] \\ \vdots\Phi_1 \\ F(b)\end{array}}{a = b}\,{=}\mathsf{I} \qquad A(a)}{A(b)}\,{=}\mathsf{E}
\quad \rightsquigarrow \quad
\begin{array}{c} A(a) \\ \vdots\Phi_1^* \\ A(b) \end{array}
$$

$$
\begin{array}{c}
F(a) \\
\vdots\,\Phi_1 \\
F(b) \qquad \Gamma \\
\vdots\,\Phi_2 \\
C
\end{array}
\quad \rightsquigarrow \quad
\begin{array}{c}
=\mathrm{E}\ \dfrac{a = b \qquad F(a)}{F(b)} \qquad \Gamma \\
\vdots\,\Phi_2 \\
C
\end{array}
$$

Note: $\begin{array}{c} F(a) \\ \vdots\,\Phi_1 \\ F(b) \end{array}$ is the direct ground for $a = b$, and indeed it is not used in the second proof tree.

$$\dfrac{\begin{array}{c} F(a) \\ \vdots\Phi_1 \\ F(b) \end{array} \qquad \Gamma}{\begin{array}{c} \vdots\Phi_2 \\ C \end{array}} \quad \rightsquigarrow \quad {=}\mathrm{E}\,\dfrac{a = b \qquad F(a)}{F(b)} \qquad \Gamma \quad \begin{array}{c} \vdots\Phi_2 \\ C \end{array}$$

Note: $\begin{array}{c} F(a) \\ \vdots\Phi_1 \\ F(b) \end{array}$ is the direct ground for $a = b$, and indeed it is not used in the second proof tree.

Stability of =-rules

$$\begin{array}{ccc} F(a) & & \\ \vdots\,\Phi_1 & & \\ F(b) & & \Gamma \\ & \vdots\,\Phi_2 & \\ & C & \end{array} \quad \rightsquigarrow \quad \begin{array}{cc} =_E \dfrac{a = b \qquad F(a)}{F(b)} & \Gamma \\ \vdots\,\Phi_2 & \\ C & \end{array}$$

Note: $\begin{array}{c} F(a) \\ \vdots\,\Phi_1 \\ F(b) \end{array}$ is the direct ground for $a = b$, and indeed it is not used in the second proof tree.

Stability of $=$-rules

$$\dfrac{\dfrac{F(a)}{\vdots\,\Phi_1}}{F(b)} \qquad \Gamma \qquad\qquad \dfrac{}{\vdots\,\Phi_2 \atop C}$$

$$\rightsquigarrow$$

$$\dfrac{\dfrac{a = b \qquad F(a)}{F(b)}\,{=}\mathsf{E} \qquad \Gamma}{\vdots\,\Phi_2 \atop C}$$

Note: $\begin{matrix} F(a) \\ \vdots\,\Phi_1 \\ F(b) \end{matrix}$ is the direct ground for $a = b$, and indeed it is not used in the second proof tree.

$$
\begin{array}{c}
F(a) \\
\vdots \Phi_1 \\
F(b) \qquad \Gamma \\
\vdots \Phi_2 \\
C
\end{array}
\quad \rightsquigarrow \quad
\begin{array}{c}
=\mathsf{E}\,\dfrac{a = b \qquad F(a)}{F(b)} \qquad \Gamma \\
\vdots \Phi_2 \\
C
\end{array}
$$

Note: $\begin{array}{c} F(a) \\ \vdots \Phi_1 \\ F(b) \end{array}$ is the direct ground for $a = b$, and indeed it is not used in the second proof tree.

Infinitary Version: Harmony

Rules

$$
\dfrac{\begin{matrix}[A_1(a)] \\ \vdots\Phi_1 \\ A_1(b)\end{matrix} \qquad \begin{matrix}[A_2(a)] \\ \vdots\Phi_1 \\ A_2(b)\end{matrix} \qquad \cdots}{a = b}\;{=}\mathsf{I}\infty
\qquad
\dfrac{a = b \qquad A_1(a)}{A_1(b)}\;{=}\mathsf{E}_1
\qquad
\dfrac{a = b \qquad A_2(a)}{A_2(b)}\;{=}\mathsf{E}_2
\qquad \cdots
$$

Harmony

$$
\dfrac{\dfrac{\begin{matrix}[A_1(a)] \\ \vdots\Phi_1 \\ A_1(b)\end{matrix} \qquad \begin{matrix}[A_2(a)] \\ \vdots\Phi_1 \\ A_2(b)\end{matrix} \qquad \cdots}{a = b}\;{=}\mathsf{I}\infty \qquad A_i(a)}{A_i(b)}\;{=}\mathsf{E}_i
\quad\rightsquigarrow\quad
\begin{matrix}[A_i(a)] \\ \vdots\Phi_1 \\ A_i(b)\end{matrix}
$$

Rules

$$
=\mathsf{I}\infty \; \dfrac{\begin{array}{ccc} [A_1(a)] & [A_2(a)] & \\ \vdots\Phi_1 & \vdots\Phi_1 & \\ A_1(b) & A_2(b) & \cdots \end{array}}{a = b}
\qquad
=\mathsf{E}_1 \; \dfrac{a = b \qquad A_1(a)}{A_1(b)}
\qquad
=\mathsf{E}_2 \; \dfrac{a = b \qquad A_2(a)}{A_2(b)}
\qquad \cdots
$$

Harmony

$$
=\mathsf{E}_i \; \dfrac{=\mathsf{I}\infty \; \dfrac{\begin{array}{ccc} [A_1(a)] & [A_2(a)] & \\ \vdots\Phi_1 & \vdots\Phi_1 & \\ A_1(b) & A_2(b) & \cdots \end{array}}{a = b} \qquad A_i(a)}{A_i(b)}
\quad \rightsquigarrow \quad
\begin{array}{c} [A_i(a)] \\ \vdots\Phi_1 \\ A_i(b) \end{array}
$$

Infinitary Version: Harmony

Rules

$$
{=}\mathrm{I}\infty\ \frac{\begin{array}{c}[A_1(a)]\\ \vdots\Phi_1\\ A_1(b)\end{array}\qquad \begin{array}{c}[A_2(a)]\\ \vdots\Phi_1\\ A_2(b)\end{array}\qquad \cdots}{a = b}
\qquad
{=}\mathrm{E}_1\ \frac{a = b \qquad A_1(a)}{A_1(b)}
\qquad
{=}\mathrm{E}_2\ \frac{a = b \qquad A_2(a)}{A_2(b)}
\qquad \cdots
$$

Harmony

$$
{=}\mathrm{E}_i\ \frac{{=}\mathrm{I}\infty\ \dfrac{\begin{array}{c}[A_1(a)]\\ \vdots\Phi_1\\ A_1(b)\end{array}\qquad \begin{array}{c}[A_2(a)]\\ \vdots\Phi_1\\ A_2(b)\end{array}\qquad \cdots}{a = b} \qquad A_i(a)}{A_i(b)}
\quad \rightsquigarrow \quad
\begin{array}{c}[A_i(a)]\\ \vdots\Phi_1\\ A_i(b)\end{array}
$$

Infinitary Version: Harmony

Rules

$$=\mathsf{I}\infty\ \dfrac{\begin{array}{c}[A_1(a)]\\ \vdots\Phi_1\\ A_1(b)\end{array}\qquad \begin{array}{c}[A_2(a)]\\ \vdots\Phi_1\\ A_2(b)\end{array}\qquad \cdots}{a=b}
\qquad\qquad
=\mathsf{E}_1\ \dfrac{a=b \qquad A_1(a)}{A_1(b)}
\qquad\qquad
=\mathsf{E}_2\ \dfrac{a=b \qquad A_2(a)}{A_2(b)}
\qquad\qquad \cdots$$

Harmony

$$=\mathsf{E}_i\ \dfrac{=\mathsf{I}\infty\ \dfrac{\begin{array}{c}[A_1(a)]\\ \vdots\Phi_1\\ A_1(b)\end{array}\qquad \begin{array}{c}[A_2(a)]\\ \vdots\Phi_1\\ A_2(b)\end{array}\qquad \cdots}{a=b} \qquad\qquad A_i(a)}{A_i(b)}
\qquad \rightsquigarrow \qquad
\begin{array}{c}[A_i(a)]\\ \vdots\Phi_1\\ A_i(b)\end{array}$$

Rules

$$
\dfrac{\begin{array}{c}[A_1(a)]\\ \vdots\Phi_1\\ A_1(b)\end{array}\qquad\begin{array}{c}[A_2(a)]\\ \vdots\Phi_1\\ A_2(b)\end{array}\qquad \cdots}{a=b}\;{=}\mathsf{I}\infty
\qquad
\dfrac{a=b\qquad A_1(a)}{A_1(b)}\;{=}\mathsf{E}_1
\qquad
\dfrac{a=b\qquad A_2(a)}{A_2(b)}\;{=}\mathsf{E}_2
\qquad \cdots
$$

Harmony

$$
\dfrac{\dfrac{\begin{array}{c}[A_1(a)]\\ \vdots\Phi_1\\ A_1(b)\end{array}\qquad\begin{array}{c}[A_2(a)]\\ \vdots\Phi_1\\ A_2(b)\end{array}\qquad \cdots}{a=b}\;{=}\mathsf{I}\infty \qquad A_i(a)}{A_i(b)}\;{=}\mathsf{E}_i
\quad\rightsquigarrow\quad
\begin{array}{c}[A_i(a)]\\ \vdots\Phi_1\\ A_i(b)\end{array}
$$

Infinitary Version: Harmony

Rules

$$
\dfrac{\begin{array}{c}[A_1(a)] \\ \vdots\Phi_1 \\ A_1(b)\end{array} \qquad \begin{array}{c}[A_2(a)] \\ \vdots\Phi_1 \\ A_2(b)\end{array} \qquad \cdots}{a = b}\;{=}\mathsf{I}\infty
\qquad
\dfrac{a = b \qquad A_1(a)}{A_1(b)}\;{=}\mathsf{E}_1
\qquad
\dfrac{a = b \qquad A_2(a)}{A_2(b)}\;{=}\mathsf{E}_2
\qquad \cdots
$$

Harmony

$$
\dfrac{\dfrac{\begin{array}{c}[A_1(a)] \\ \vdots\Phi_1 \\ A_1(b)\end{array} \qquad \begin{array}{c}[A_2(a)] \\ \vdots\Phi_1 \\ A_2(b)\end{array} \qquad \cdots}{a = b}\;{=}\mathsf{I}\infty \qquad A_i(a)}{A_i(b)}\;{=}\mathsf{E}_i
\qquad \rightsquigarrow \qquad
\begin{array}{c}[A_i(a)] \\ \vdots\Phi_1 \\ A_i(b)\end{array}
$$

Rules

$$=\mathsf{I}\infty \frac{\begin{array}{ccc}[A_1(a)] & [A_2(a)] & \\ \vdots\Phi_1 & \vdots\Phi_1 & \\ A_1(b) & A_2(b) & \cdots\end{array}}{a = b} \qquad =\mathsf{E}_1 \frac{a = b \quad A_1(a)}{A_1(b)} \qquad =\mathsf{E}_2 \frac{a = b \quad A_2(a)}{A_2(b)} \qquad \cdots$$

Inverse Harmony

$$\begin{array}{cccc} A_1(a) & A_2(a) & & \\ \vdots\Phi_1 & \vdots\Phi_2 & & \\ A_1(b) & A_2(b) & \cdots & \Gamma \\ & \vdots\Psi & & \\ & C & & \end{array} \quad \rightsquigarrow \quad \begin{array}{cccc} =\mathsf{E}\dfrac{a = b \quad A_1(a)}{A_1(b)} & =\mathsf{E}\dfrac{a = b \quad A_2(a)}{A_2(b)} & \cdots & \Gamma \\ & \vdots\Psi & & \\ & C & & \end{array}$$

Note: $\begin{array}{c} A_i(a) \\ \vdots\Phi_i \\ A_i(b) \end{array}$ are the direct grounds for $a = b$, and indeed they are not used in the second proof tree.

Infinitary Version: Stability

Rules

$$=\mathsf{I}\infty\;\frac{\begin{array}{c}[A_1(a)]\\ \vdots\Phi_1\\ A_1(b)\end{array}\qquad \begin{array}{c}[A_2(a)]\\ \vdots\Phi_1\\ A_2(b)\end{array}\qquad \cdots}{a=b}
\qquad\qquad
=\mathsf{E}_1\;\frac{a=b \qquad A_1(a)}{A_1(b)}
\qquad\qquad
=\mathsf{E}_2\;\frac{a=b \qquad A_2(a)}{A_2(b)}
\qquad\qquad \cdots$$

Inverse Harmony

$$\begin{array}{c}\begin{array}{c}A_1(a)\\ \vdots\Phi_1\\ A_1(b)\end{array}\quad \begin{array}{c}A_2(a)\\ \vdots\Phi_2\\ A_2(b)\end{array}\quad \cdots \quad \Gamma\\ \vdots\Psi\\ C\end{array}
\qquad \rightsquigarrow \qquad
\begin{array}{c}=\mathsf{E}\;\dfrac{a=b \qquad A_1(a)}{A_1(b)}\quad =\mathsf{E}\;\dfrac{a=b \qquad A_2(a)}{A_2(b)}\quad \cdots \quad \Gamma\\ \vdots\Psi\\ C\end{array}$$

Note: $\begin{array}{c}A_i(a)\\ \vdots\Phi_i\\ A_i(b)\end{array}$ are the direct grounds for $a = b$, and indeed they are not used in the second proof tree.

Rules

$$
=\mathsf{I}\infty\ \dfrac{\begin{matrix}[A_1(a)] \\ \vdots\Phi_1 \\ A_1(b)\end{matrix} \qquad \begin{matrix}[A_2(a)] \\ \vdots\Phi_1 \\ A_2(b)\end{matrix} \qquad \cdots}{a = b}
\qquad
=\mathsf{E}_1\ \dfrac{a = b \qquad A_1(a)}{A_1(b)}
\qquad
=\mathsf{E}_2\ \dfrac{a = b \qquad A_2(a)}{A_2(b)}
\qquad \cdots
$$

Inverse Harmony

$$
\begin{matrix}
\begin{matrix}A_1(a) \\ \vdots\Phi_1 \\ A_1(b)\end{matrix} \quad \begin{matrix}A_2(a) \\ \vdots\Phi_2 \\ A_2(b)\end{matrix} \quad \cdots \quad \Gamma \\
\vdots\Psi \\
C
\end{matrix}
\quad \rightsquigarrow \quad
\begin{matrix}
=\mathsf{E}\ \dfrac{a = b \qquad A_1(a)}{A_1(b)} \quad =\mathsf{E}\ \dfrac{a = b \qquad A_2(a)}{A_2(b)} \quad \cdots \quad \Gamma \\
\vdots\Psi \\
C
\end{matrix}
$$

Note: $\begin{matrix}A_i(a) \\ \vdots\Phi_i \\ A_i(b)\end{matrix}$ are the direct grounds for $a = b$, and indeed they are not used in the second proof tree.

Infinitary Version: Stability

Rules

$$=\mathsf{I}\infty\ \dfrac{\begin{matrix}[A_1(a)] & [A_2(a)] & \\ \vdots\Phi_1 & \vdots\Phi_1 & \\ A_1(b) & A_2(b) & \cdots\end{matrix}}{a=b} \qquad =\mathsf{E}_1\ \dfrac{a=b \quad A_1(a)}{A_1(b)} \qquad =\mathsf{E}_2\ \dfrac{a=b \quad A_2(a)}{A_2(b)} \qquad \cdots$$

Inverse Harmony

$$\begin{matrix}A_1(a) & A_2(a) & & \\ \vdots\Phi_1 & \vdots\Phi_2 & & \\ A_1(b) & A_2(b) & \cdots & \Gamma \\ & \vdots\Psi & & \\ & C & & \end{matrix} \quad \rightsquigarrow \quad \begin{matrix}=\mathsf{E}\ \dfrac{a=b \quad A_1(a)}{A_1(b)} & =\mathsf{E}\ \dfrac{a=b \quad A_2(a)}{A_2(b)} & \cdots & \Gamma \\ & \vdots\Psi & & \\ & C & & \end{matrix}$$

Note: $\begin{matrix}A_i(a) \\ \vdots\Phi_i \\ A_i(b)\end{matrix}$ are the direct grounds for $a = b$, and indeed they are not used in the second proof tree.

Infinitary Version: Stability

Rules

$$=\mathsf{I}\infty\ \dfrac{\begin{array}{c}[A_1(a)]\\ \vdots\Phi_1\\ A_1(b)\end{array}\qquad\begin{array}{c}[A_2(a)]\\ \vdots\Phi_1\\ A_2(b)\end{array}\qquad\cdots}{a=b}\qquad\qquad =\mathsf{E}_1\ \dfrac{a=b\qquad A_1(a)}{A_1(b)}\qquad\qquad =\mathsf{E}_2\ \dfrac{a=b\qquad A_2(a)}{A_2(b)}\qquad\qquad\cdots$$

Inverse Harmony

$$\begin{array}{c}\begin{array}{cccc}A_1(a) & A_2(a) & & \\ \vdots\Phi_1 & \vdots\Phi_2 & & \\ A_1(b) & A_2(b) & \cdots & \Gamma\end{array}\\ \vdots\Psi\\ C\end{array}\qquad\rightsquigarrow\qquad\begin{array}{c}=\mathsf{E}\ \dfrac{a=b\quad A_1(a)}{A_1(b)}\quad =\mathsf{E}\ \dfrac{a=b\quad A_2(a)}{A_2(b)}\quad\cdots\quad\Gamma\\ \vdots\Psi\\ C\end{array}$$

Note: $\begin{array}{c}A_i(a)\\ \vdots\Phi_i\\ A_i(b)\end{array}$ are the direct grounds for $a=b$, and indeed they are not used in the second proof tree.

Infinitary Version: Stability

Rules

$$
{=}\mathsf{I}\infty\ \dfrac{\begin{matrix}[A_1(a)] \\ \vdots\Phi_1 \\ A_1(b)\end{matrix} \qquad \begin{matrix}[A_2(a)] \\ \vdots\Phi_1 \\ A_2(b)\end{matrix} \qquad \cdots}{a = b}
\qquad
{=}\mathsf{E}_1\ \dfrac{a = b \qquad A_1(a)}{A_1(b)}
\qquad
{=}\mathsf{E}_2\ \dfrac{a = b \qquad A_2(a)}{A_2(b)}
\qquad \cdots
$$

Inverse Harmony

$$
\begin{matrix}
A_1(a) & A_2(a) & & \\
\vdots\Phi_1 & \vdots\Phi_2 & & \\
A_1(b) & A_2(b) & \cdots & \Gamma \\
 & \vdots\Psi & & \\
 & C & &
\end{matrix}
\qquad \rightsquigarrow \qquad
\begin{matrix}
{=}\mathsf{E}\ \dfrac{a = b \qquad A_1(a)}{A_1(b)} \quad {=}\mathsf{E}\ \dfrac{a = b \qquad A_2(a)}{A_2(b)} \quad \cdots \quad \Gamma \\
\vdots\Psi \\
C
\end{matrix}
$$

Note: $\begin{matrix}A_i(a) \\ \vdots\Phi_i \\ A_i(b)\end{matrix}$ are the direct grounds for $a = b$, and indeed they are not used in the second proof tree.

Rules

$$
\dfrac{\begin{array}{c}[A_1(a)]\\ \vdots\Phi_1\\ A_1(b)\end{array} \qquad \begin{array}{c}[A_2(a)]\\ \vdots\Phi_1\\ A_2(b)\end{array} \qquad \cdots}{a = b}\ {=}\mathsf{I}\infty
\qquad
\dfrac{a = b \qquad A_1(a)}{A_1(b)}\ {=}\mathsf{E}_1
\qquad
\dfrac{a = b \qquad A_2(a)}{A_2(b)}\ {=}\mathsf{E}_2
\qquad \cdots
$$

Inverse Harmony

$$
\begin{array}{c}
\begin{array}{c}A_1(a)\\ \vdots\Phi_1\\ A_1(b)\end{array} \quad \begin{array}{c}A_2(a)\\ \vdots\Phi_2\\ A_2(b)\end{array} \quad \cdots \quad \Gamma \\
\vdots\Psi \\
C
\end{array}
\quad\rightsquigarrow\quad
\begin{array}{c}
\dfrac{a = b \qquad A_1(a)}{A_1(b)}\ {=}\mathsf{E} \quad \dfrac{a = b \qquad A_2(a)}{A_2(b)}\ {=}\mathsf{E} \quad \cdots \quad \Gamma \\
\vdots\Psi \\
C
\end{array}
$$

Note: $\begin{array}{c}A_i(a)\\ \vdots\Phi_i\\ A_i(b)\end{array}$ are the direct grounds for $a = b$, and indeed they are not used in the second proof tree.

Fractions

$$\div\mathsf{I}\ \frac{n \times s = l \times m}{n/m = l/s} \qquad \div\mathsf{E}\ \frac{n/m = l/s}{n \times s = l \times m}$$

Note: $a/b + c/d = e/f$ is not well formed in **BA**.

Atomic completeness: for every atomic sentence E, $\vdash_{\mathbf{BA}} E$ or $\vdash_{\mathbf{BA}} \neg E$.

Fractions

$$\div\mathsf{I}\ \frac{n \times s = l \times m}{n/m = l/s} \qquad \div\mathsf{E}\ \frac{n/m = l/s}{n \times s = l \times m}$$

Note: $a/b + c/d = e/f$ is not well formed in **BA**.

Atomic completeness: for every atomic sentence E, $\vdash_{\mathbf{BA}} E$ or $\vdash_{\mathbf{BA}} \neg E$.

Fractions

$$\div\mathsf{I}\ \frac{n \times s = l \times m}{n/m = l/s} \qquad \div\mathsf{E}\ \frac{n/m = l/s}{n \times s = l \times m}$$

Note: $a/b + c/d = e/f$ is not well formed in **BA**.

Atomic completeness: for every atomic sentence E, $\vdash_{\mathbf{BA}} E$ or $\vdash_{\mathbf{BA}} \neg E$.

Fractions

$$\div\text{I}\ \frac{n \times s = l \times m}{n/m = l/s} \qquad \div\text{E}\ \frac{n/m = l/s}{n \times s = l \times m}$$

Note: $a/b + c/d = e/f$ is not well formed in **BA**.

Atomic completeness: for every atomic sentence E, $\vdash_{\mathbf{BA}} E$ or $\vdash_{\mathbf{BA}} \neg E$.

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\begin{matrix}\vdots\\ 1/2=2/4\end{matrix}\qquad \dfrac{\dfrac{[F((1+1)/(2+3))]^1}{(1+1)/(2+3)=(1+1)/(2+3)}\,{=}\mathrm{I}_1}{(1/2)?(1/3)=(1+1)/(2+3)}\,\mathrm{I}?}{(2/4)?(1/3)=(1+1)/(2+3)}\,{=}\mathrm{E}}{(2+1)/(4+3)=(1+1)/(2+3)}\,\mathrm{E}? \qquad \begin{matrix}\vdots\\ 1+1=2\end{matrix}}{(2+1)/(4+3)=2/(2+3)}\,{=}\mathrm{E} \qquad \begin{matrix}\vdots\\ 2+3=5\end{matrix}}{(2+1)/(4+3)=2/5}\,{=}\mathrm{E} \qquad \begin{matrix}\vdots\\ 2+1=3\end{matrix}}{3/(4+3)=2/5}\,{=}\mathrm{E} \qquad \begin{matrix}\vdots\\ 4+3=7\end{matrix}}{3/7=2/5}\,{=}\mathrm{E}}{3\times 5=2\times 7}\,{\div}\mathrm{E}}{15=14}
$$

$$
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$$

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- **BA** is stable;
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Thanks for your attention!

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