Is logical disagreement possible in inferentialism?

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Identity and disagreement

Alice: Mum is blonde!

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Bob: Mum has brown hair!

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Why?

Alice: Mum is blonde!

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They disagree if and only if they are brother and sister.

Why?

Because otherwise they do not speak of the same person!

Logical disagreement

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Intuitionistic logic: $\nvDash p \vee \neg p$

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Classical logic: $\models p \vee \neg p$

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In order to disagree, **two logics have to speak about the same logical terms.**

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In order to disagree, **two logics have to speak about the same logical terms.**

They disagree

- $\models_k p \vee \neg p$
- $\not\models_i p \vee \neg p$

They don't disagree

- $\models_k p \vee_k \neg_k p$
- $\not\models_i p \vee_i \neg_i p$

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Both truth tables and rules of inference determine univocally the class of logical laws for a class of logical terms.

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Both truth tables and rules of inference determine univocally the class of logical laws for a class of logical terms.

In order to be the same, two logical terms should have the same truth tables (according to realism) or they should have the same rules (according to antirealism). So **if they are the same, they validate the same logical laws.**

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'Change of logic, change of subject.' (Quine, *Philosophy of Logic*, 1986, chapter *Deviant logics*.)

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- $\vDash_k p \vee_k \neg_k p$
- $\nvDash_i p \vee_i \neg_i p$

A realist solution

Meaning and behaviour in realism

$A \vee B$	T	F
T	T	T
F	T	F

A	$\neg A$
T	F
F	T

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If we assume **bivalence** (every sentence is true or false) they lead to **classical logic**.

BUT if we do not assume bivalence they **they can characterise trivalent connectives in an incomplete way!**

They also suit intuitionistic connectives if we do not accept bivalence. (McDowell, *Meaning, bivalence, and verificationism*, in Gareth Evans ed., *Truth and meaning: essays in semantics*, 1976, pp. 42-66.)

K3

K3

A	$\neg A$
T	F
U	U
F	T

$A \vee B$	T	U	F
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$\Gamma \vDash_{K3} \Delta \ \ sse \ \bar{\forall} v((\bar{\forall}\gamma_{\in\Gamma} v(\gamma) = T) \Rightarrow (\bar{\exists}\delta_{\in\Delta} v(\delta) = T))$

designated value: T

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If we assume **gaps** in truth values, they lead to **Kleene's logic**.

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If we assume **gaps** in truth values, they lead to **Kleene's logic**.

If we assume **bivalence**, they lead to **classical logic**.

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If we assume **gaps** in truth values, they lead to **Kleene's logic**.

If we assume **bivalence**, they lead to **classical logic**.

Idea: **K** is essentially **K3** where there are no gaps!

Beall & Restall solution

The difference between $\vee_k$ and $\vee_{k3}$ is not a difference in truth tables, so they have the **same meaning**.

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The difference between $\vee_k$ and $\vee_{k3}$ is not a difference in truth tables, so they have the **same meaning**. It is just a **metaphysical** difference that determines a difference in **behaviour**.

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In general:

Generalized Tarski Thesis (GTT): An argument is **valid**$_x$ if and only if, in every **case**$_x$ in which the premises are true, so is the conclusion.(Beall & Restall, *Logical Pluralism*, 2006, p. 29.)

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Different sets of cases detect different logics:

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Different sets of cases detect different logics: cases can have gaps in truth values, gluts in truth values, etc.

A logical realist can reject the point **2** of the puzzle: **two logical terms can be the same also if they validate different logical laws.**

An antirealist solution

Nothing external

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According to antirealism, the meaning of a logical term consists in its inferential role.

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So a difference in inferential behaviour is a difference in meaning.

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So we can not reject point **2** of the puzzle!

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So we can not reject point **2** of the puzzle!

We need another solution!

A case study

Two factions in proof-theoretic semantics:

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(a **rational reconstruction** of the debate)

Two factions in proof-theoretic semantics:

Prawitz & Dummett: *tertium non datur* does not hold!

Boričić, Read & Milne: *tertium non datur* holds!

(a **rational reconstruction** of the debate)

We will see that **their disagreement is not only apparent!**
They don't talk past each other.

Proof-theoretic semantics

Natural deduction as a **theory of meaning**.

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- **Introduction** rules are **meaning conferring**;

Natural deduction as a **theory of meaning**.

- **Introduction** rules are **meaning conferring**;
- **Elimination** rules are **justified** by I-rules.

Natural deduction as a **theory of meaning**.

- **Introduction** rules are **meaning conferring**;
- **Elimination** rules are **justified** by I-rules.

"The introductions represent, as it were, the 'definitions' of the symbols concerned, and the eliminations are no more, in the final analysis, than the consequences of these definitions. This fact may be expressed as follows: In eliminating a symbol, we may use the formula with whose terminal symbol we are dealing only 'in the sense afforded it by the introduction of that symbol'." (Gentzen, *Investigation into Logical Deduction*, 1934/35, 5.13)

Harmony

The justification of E-rules.

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- E-rules can not be derived from I-rules;

The justification of E-rules.

- E-rules can not be derived from I-rules;
- E-rules do not outstrip the I-rules if, when we have the **major premise of an E-rule derived using an I-rule**, then we have an **avoidable detour**;

The justification of E-rules.

- E-rules can not be derived from I-rules;
- E-rules do not outstrip the I-rules if, when we have the **major premise of an E-rule derived using an I-rule**, then we have an **avoidable detour**;
- Otherwise the E-rules are not justified!

Prawitz & Dummett

$$\frac{A}{A \vee B} \vee I$$

$$\frac{B}{A \vee B} \vee I$$

$$\frac{\begin{array}{c}[A] \\ \vdots \\ B\end{array}}{A \supset B} \supset I$$

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$$\frac{\begin{array}{c}[A] \\ \vdots \\ B\end{array}}{A \supset B} \supset I$$

$$\frac{A \vee B \quad \begin{array}{c}[A] \\ \vdots \\ C\end{array} \quad \begin{array}{c}[B] \\ \vdots \\ C\end{array}}{C} \vee E$$

$$\frac{A \supset B \quad A}{B} \supset E$$

$$\frac{A}{A \vee B}\ \vee I$$

$$\frac{B}{A \vee B}\ \vee I$$

$$\frac{\begin{matrix}[A]\\ \vdots\\ B\end{matrix}}{A \supset B}\ \supset I$$

$$\frac{A \vee B \quad \begin{matrix}[A]\\ \vdots\\ C\end{matrix} \quad \begin{matrix}[B]\\ \vdots\\ C\end{matrix}}{C}\ \vee E$$

$$\frac{A \supset B \quad A}{B}\ \supset E$$

$$\frac{\bot}{A}\ \bot E$$

Harmony of $\vee$

Harmony of ∨

$$
\dfrac{\dfrac{\begin{matrix}\vdots\Phi_1\\ A\end{matrix}}{A \vee B}\,\vee I \qquad \begin{matrix}[A]\\ \vdots\Phi_2\\ C\end{matrix} \qquad \begin{matrix}[B]\\ \vdots\Phi_3\\ C\end{matrix}}{C}\,\vee E
$$

$$
\vdots\Phi_4
$$

Harmony of $\vee$

$$\dfrac{\dfrac{\begin{matrix}\vdots\,\Phi_1\\ A\end{matrix}}{A \vee B}\,\vee I \qquad \begin{matrix}[A]\\ \vdots\,\Phi_2\\ C\end{matrix} \qquad \begin{matrix}[B]\\ \vdots\,\Phi_3\\ C\end{matrix}}{\begin{matrix}C\\ \vdots\,\Phi_4\end{matrix}}\,\vee E \qquad\qquad \rightsquigarrow$$

Harmony of ∨

$$
\begin{array}{c}
\dfrac{\dfrac{\begin{matrix}\vdots\,\Phi_1\\ A\end{matrix}}{A \vee B}\,\vee I \qquad \begin{matrix}[A]\\ \vdots\,\Phi_2\\ C\end{matrix} \qquad \begin{matrix}[B]\\ \vdots\,\Phi_3\\ C\end{matrix}}{C}\,\vee E \\
\vdots\,\Phi_4
\end{array}
\quad \rightsquigarrow \quad
\begin{array}{c}
\vdots\,\Phi_1\\
[A]\\
\vdots\,\Phi_2\\
C\\
\vdots\,\Phi_4
\end{array}
$$

Harmony of ⊃

Harmony of $\supset$

$$\cfrac{\cfrac{\begin{array}{c}[A]\\ \vdots\,\Phi_1\\ B\end{array}}{A \supset B}\supset I \qquad \begin{array}{c}\vdots\,\Phi_2\\ A\end{array}}{\begin{array}{c}B\\ \vdots\,\Phi_3\end{array}}\supset E$$

Harmony of $\supset$

$$\dfrac{\dfrac{\begin{matrix}[A] \\ \vdots \Phi_1 \\ B\end{matrix}}{A \supset B} \supset I \qquad \begin{matrix}\vdots \Phi_2 \\ A\end{matrix}}{B} \supset E$$

$$\vdots \Phi_3$$

$$\rightsquigarrow$$

Harmony of ⊃

$$\dfrac{\dfrac{\begin{array}{c}[A] \\ \vdots \Phi_1 \\ B\end{array}}{A \supset B} \supset I \qquad \begin{array}{c}\vdots \Phi_2 \\ A\end{array}}{B} \supset E$$
$$\vdots \Phi_3$$

$$\rightsquigarrow$$

$$\begin{array}{c}\vdots \Phi_2 \\ [A] \\ \vdots \Phi_1 \\ B \\ \vdots \Phi_3\end{array}$$

Tonk

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$$Itonk\ \frac{A}{AtonkB} \qquad Etonk\ \frac{AtonkB}{B}$$

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$$\begin{array}{l} Itonk \\ Etonk \end{array} \frac{\dfrac{A}{AtonkB}}{B}$$

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$$Itonk\ \frac{A}{AtonkB} \qquad Etonk\ \frac{AtonkB}{B}$$

$$\dfrac{Itonk\ \dfrac{A}{AtonkB}}{B}\,Etonk \qquad \rightsquigarrow$$

Tonk

$$Itonk\ \frac{A}{AtonkB} \qquad Etonk\ \frac{AtonkB}{B}$$

$$\begin{array}{r l} Itonk & \dfrac{A}{AtonkB} \\ Etonk & \dfrac{}{B} \end{array} \quad \rightsquigarrow \quad ?$$

Tonk

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- *Tonk* is not an harmonious connective, and indeed it leads to triviality in standard logical systems.

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- *Tonk* is not an harmonious connective, and indeed it leads to triviality in standard logical systems.

(Prior, *the runabout inference ticket*, Analysis (21), 1961, pp. 38-39)

Rejection of *tertium non datur*

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(Prawitz, *Towards a foundation of general proof theory*, in P. Suppes et al (ed), *Logic, Methodology and Philosophy of Science IV*, 1973, pp. 225-50.)

(Dummett, *The Logical Basis of Metaphysics*, 1991.)

$$\frac{A, \Gamma}{A \vee B, \Gamma} \vee I$$

$$\frac{B, \Gamma}{A \vee B, \Gamma} \vee I$$

$$\frac{\begin{array}{c}[A]\\ \vdots \\ B, \Gamma\end{array}}{A \supset B, \Gamma} \supset I$$

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$$\frac{A \supset B, \Gamma \quad A, \Theta \quad \begin{matrix}[B] \\ \vdots \\ \Delta\end{matrix}}{\Gamma, \Delta, \Theta} \supset E$$

$$\frac{A, \Gamma}{A \vee B, \Gamma} \vee I$$

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$$\frac{A \supset B, \Gamma \quad A, \Theta \quad \begin{matrix}[B] \\ \vdots \\ \Delta\end{matrix}}{\Gamma, \Delta, \Theta} \supset E$$

$$\frac{\bot, \Delta}{A, \Delta} \bot E \qquad \frac{\Delta}{A, \Delta} \text{Weakening} \qquad \frac{A, A, \Delta}{A, \Delta} \text{Contraction}$$

Acceptance of *tertium non datur*

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Boričić & Read: **We can prove *tertium non datur* using harmonious rules!**

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$$\dfrac{\dfrac{[A]^1}{A, \bot}\text{ Weakening}}{A, A \supset \bot} \supset I_1$$

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(Boričić, *On sequence-conclusion natural deduction systems*, Journal of Philosophical Logic (14), 1985, pp. 359-377.)
(Read, *Harmony and autonomy in classical logic*, Journal of Philosophical Logic (29), 2000, pp. 123-54.)

Objections to Boričić & Read's solution

- **Multiple conclusions are not commonly used!**

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  [...] the rarity, to the point of extinction, of naturally occurring multiple-conclusion arguments has always been the reason why mainstream logicians have dismissed multiple-conclusion logic as little more than a curiosity. (Rumfitt, Knowledge by deduction, Grazer Philosophische Studien (77), 2008, pp. 61-84)

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  [. . .] this smuggles in non-constructivity through the back door. (Tennant, The Taming of the True, 1997)

- **Multiple conclusions are just disjunction in disguise!**

  [. . .] in a succedent comprising more than one sentence, the sentences are connected disjunctively; and it is not possible to grasp the sense of such a connection otherwise than by learning the meaning of the constant 'or'. (Dummett, Ibidem, p. 187)

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Disagreements: constructivity.

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A theory of meaning should use a constructive logic? This is so controversial, I will neglect discussing it!

Disagreements: multiple conclusions and disjunction.

Milne's formulates classical logic with:

$$
I\supset_{Mln}\ \frac{\begin{array}{c}[A]\\ \vdots\\ B\{\vee D\}\end{array}}{(A\supset B)\{\vee D\}}
\qquad
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(Milne, *Harmony, Purity, Simplicity and a "Seemingly Magical Fact"*, The Monist (85), 2002, pp. 498-534)

Repetita iuvant

$$
\dfrac{\dfrac{\begin{array}{c}[A] \\ \vdots\,\Phi_1 \\ B\end{array}}{A \supset B}\supset I \qquad \begin{array}{c}\vdots\,\Phi_2 \\ A\end{array}}{B}\supset E
\qquad
\begin{array}{c}\vdots\,\Phi_3\end{array}
\quad\rightsquigarrow\quad
\begin{array}{c}\vdots\,\Phi_2 \\ [A] \\ \vdots\,\Phi_1 \\ B \\ \vdots\,\Phi_3\end{array}
$$

Harmony of $\supset_{Mln}$

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$$\dfrac{\dfrac{\begin{matrix}[A]^1\\ \vdots\,\Phi_1\\ B\vee C\end{matrix}}{(A\supset B)\vee C}\supset I \qquad \begin{matrix}\dfrac{[A\supset B]^2 \qquad A}{B}\supset E\\ \vdots\,\Phi_2\\ D\end{matrix} \qquad \begin{matrix}[C]^2\\ \vdots\,\Phi_3\\ D\end{matrix}}{\begin{matrix}D\\ \vdots\,\Phi_4\end{matrix}}\vee E_2$$

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$$\downarrow$$

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$$

$$
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$$

$$
\downarrow
$$

$$
\cfrac{\begin{matrix}A\\ \vdots\Phi_1\\ B\vee C\end{matrix} \qquad \begin{matrix}[B]^2\\ \vdots\Phi_2\\ D\end{matrix} \qquad \begin{matrix}[C]^2\\ \vdots\Phi_3\\ D\end{matrix}}{D}\vee E_2
$$

$$
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Can it be non-constructive?

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Essentially: Can it have multiple conclusions?

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They don't talk past each other.

Thanks for your attention!

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